



Parity, eulerian subgraphs and the Tutte polynomial

A.J. Goodall

University of Bristol, University Walk, Bristol BS8 1TW, UK

Received 6 March 2007

Available online 5 November 2007

Abstract

Identities obtained by elementary finite Fourier analysis are used to derive a variety of evaluations of the Tutte polynomial of a graph G at certain points (a, b) where $(a - 1)(b - 1) \in \{2, 4\}$. These evaluations are expressed in terms of eulerian subgraphs of G and the size of subgraphs modulo 2, 3, 4 or 6. In particular, a graph is found to have a nowhere-zero 4-flow if and only if there is a correlation between the event that three subgraphs A, B, C chosen uniformly at random have pairwise eulerian symmetric differences and the event that $\lfloor \frac{|A|+|B|+|C|}{3} \rfloor$ is even. Some further evaluations of the Tutte polynomial at points (a, b) where $(a - 1)(b - 1) = 3$ are also given that illustrate the unifying power of the methods used. The connection between results of Matiyasevich, Alon and Tarsi and Onn is highlighted by indicating how they may all be derived by the techniques adopted in this paper.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Tutte polynomial; Weight enumerator; Eulerian subgraphs; Fourier transform; Parity; Correlation

1. Introduction

Let G be a finite graph, loops and parallel edges permitted. This article continues a series of papers [6,7] using elementary finite Fourier analysis to derive evaluations of the Tutte polynomial $T(G; x, y)$ at certain points $(x, y) = (a, b)$ where $(a - 1)(b - 1) \in \{2, 3, 4\}$. Letting H_q denote the hyperbola $\{(x, y): (x - 1)(y - 1) = q\}$, the evaluations of the Tutte polynomial on H_2 and H_4 are expressed in terms of eulerian subgraphs of G and the size of subgraphs modulo 2, 3, 4 or 6, the evaluations on H_3 in terms of directed eulerian subgraphs and size modulo 3.

E-mail address: a.goodall@bristol.ac.uk.

The theorems obtained in this article are analogous to the well-known fact that the eulerian subgraphs of G all have the same size modulo 2 if G is bipartite and otherwise the number of eulerian subgraphs of odd size is equal to the number of even size. They seem however to be more elusive of explanation than this straightforward example. In particular, Theorem 20 states that a graph G has a nowhere-zero 4-flow if and only if there is a correlation between (i) the event that three subgraphs A, B, C chosen uniformly at random have pairwise eulerian symmetric differences and (ii) the event that $\lfloor \frac{|A|+|B|+|C|}{3} \rfloor$ is even. A companion to this theorem is Theorem 18 which states that G is eulerian (has a nowhere-zero 2-flow) if and only if there is a correlation between (i) the event that subgraphs A, B chosen uniformly at random have eulerian symmetric difference and (ii) the event that $\lfloor \frac{|A|+|B|}{2} \rfloor$ is even. In this language, a graph G is bipartite (has a nowhere-zero 2-tension) if and only if there is a correlation between the event that a subgraph A chosen uniformly at random is eulerian and the event that $|A|$ is even.

In Section 3 we develop a result of Onn [12] which states a parity criterion for the existence of nowhere-zero q -flows of a graph. This leads to another correlation between a parity event and an event involving eulerian subgraphs of G , this time connected to whether or not G has a proper vertex 4-colouring. Onn in his paper uses the algebraic method used by Alon and Tarsi [1–3] in their proof of a parity criterion for the existence of proper q -colourings of G in terms of eulerian subgraphs of an orientation of G .

In Section 4 we consider graphs with a cycle double cover by triangles (such as plane triangulations) and derive a criterion for the existence of a proper vertex 4-colouring of G in terms of the correlation between (i) the event that a pair of subgraphs A, B of G are eulerian and between them cover all the edges of G and (ii) the event that $|A| \equiv |B| \pmod{3}$.

In Section 5 we prove theorems of a similar character to the cited results of Alon and Tarsi, involving evaluations of the Tutte polynomial on H_3 , this time confining our attention to 4-regular graphs (such as line graphs of cubic graphs). These results stem also from the work of Matiyasevich [10], whose probabilistic restatements of the Four Colour Theorem inspired the mode of expression for the Tutte polynomial evaluations throughout this paper.

The method of proof throughout is to use identities from elementary Fourier analysis from which the interpretations of the Tutte polynomial evaluations can be extracted. The results of Section 2 (the main theorems of which are to be found in Sections 2.3.1 to 2.3.4) ultimately derive from Lemma 1, which comprises a set of identities which can be found in [11]. In the language of coding theory, these identities relate the sum of powers of coset weight enumerators of a binary code to the Hamming weight enumerator of the code. In the context of this paper, the code is the cycle space of a graph. In Sections 3, 4 and 5 the main tool is the discrete version of the Poisson summation formula (or, in the context of coding theory, the MacWilliams duality theorem for complete weight enumerators). A simultaneous generalisation of Lemma 1 and the Poisson summation formula is presented in Lemma 21; this is used in Section 5.

A more expansive exposition of the material in Sections 1.2 and 1.3 of the present article can be found in [7] but is included here for convenience. All the facts quoted without reference in Section 1.2 on Fourier transforms can be found for example in [18].

Eulerian subgraphs of G are cycles in the graphic matroid underlying G . In [9] the results of [6] are extended from graphs to matrices. In a similar way, the results of the present paper depend only on the cycle space of G and could be extended to binary matroids.

1.1. Notation and definitions

Let $G = (V, E)$ be a graph. In Section 2 we consider $\mathcal{C}_2 \subseteq \mathbb{F}_2^E$ the subspace of 2-flows (eulerian subgraphs, cycles) of G , and $\mathbb{F}_2^E/\mathcal{C}_2$ the set of cosets of \mathcal{C}_2 in the additive group \mathbb{F}_2^E . The quotient space $\mathbb{F}_2^E/\mathcal{C}_2$ is isomorphic to the orthogonal subspace \mathcal{C}_2^\perp of 2-tensions (cutsets, cocycles) of G .

The rank of G is defined by $r(G) = |V| - k(G)$, where $k(G)$ is the number of components of G , and the nullity by $n(G) = |E| - r(G)$. The subspace \mathcal{C}_2 has dimension $n(G)$ over \mathbb{F}_2 , and \mathcal{C}_2^\perp dimension $r(G)$.

To each subset $A \subseteq E$ there is a subgraph (V, A) of G obtained by deleting the edges in $E \setminus A$ from G . For short this subgraph will be referred to just by its edge set A , “the subgraph A ” meaning the graph (V, A) . A subgraph A is *eulerian* if all its vertex degrees are even. The subspace \mathcal{C}_2 of 2-flows of G may be identified with the set of eulerian subgraphs of G . For two subgraphs A, B of G the symmetric difference $A \triangle B$ corresponds to addition of the indicator vectors of A and B in \mathbb{F}_2^E . The size $|A|$ of A is equal to the Hamming weight of the indicator vector of A . Two subsets $A, B \subseteq E$ belong to the same coset of \mathcal{C}_2 if and only if $A \triangle B$ is eulerian, and this is the case if and only if the subgraphs A, B have the same degree sequence modulo 2.

The space of \mathbb{F}_4 -flows of G will be denoted by \mathcal{C}_4 . The space \mathcal{C}_4 is isomorphic to $\mathcal{C}_2 \times \mathcal{C}_2$. From this observation a graph $G = (V, E)$ has a nowhere-zero 4-flow if and only if there are two eulerian subgraphs which together cover E , whence the well-known equivalence of the Four Colour Theorem with existence of an edge covering of any given planar graph by two of its eulerian subgraphs.

For $A \subseteq E$ the rank $r(A)$ of A is defined to be the rank of the subgraph (V, A) . The *Tutte polynomial* of G is defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

The hyperbolae $H_q = \{(x, y) : (x-1)(y-1) = q\}$ for $q \in \mathbb{N}$ play a special role in the theory of the Tutte polynomial, summarised for example in [20, §3.7]. In particular, $(-1)^{r(G)} q^{k(G)} \times T(G; 1-q, 0) = P(G; q)$ is the number of proper vertex q -colourings of G and $(-1)^{n(G)} \times T(G; 0, 1-q) = F(G; q)$ is the number of nowhere-zero \mathbb{Z}_q -flows of G .

On H_2 there are the evaluations $P(G; 2) = (-1)^{r(G)} 2^{k(G)} T(G; -1, 0)$ and $F(G; 2) = (-1)^{n(G)} T(G; 0, -1)$. The Tutte polynomial on H_2 is the partition function of the Ising model of statistical physics, or, what is the same thing, the Hamming weight enumerator of the subspace \mathcal{C}_2 of 2-flows (this is van der Waerden’s eulerian expansion of the Ising model [19]). In Section 2 the Tutte polynomial at the points $(-2, \frac{1}{3})$ and $(-\frac{1}{2}, -\frac{1}{3})$ on H_2 , in addition to $(-1, 0)$ and $(0, -1)$, are given an interpretation in terms of eulerian subgraphs of G .

On H_4 we have $P(G; 4) = (-1)^{r(G)} 4^{k(G)} T(G; -3, 0)$ and $F(G; 4) = (-1)^{n(G)} T(G; 0, -3)$. Also, $T(G; -1, -1) = (-1)^{|E|} (-2)^{\dim(\mathcal{C}_2 \cap \mathcal{C}_2^\perp)}$, where $\mathcal{C}_2 \cap \mathcal{C}_2^\perp$ is the bicycle space of G [14]. The Tutte polynomial on H_4 is the partition function of the 4-state Potts model, which coincides with the Hamming weight enumerator of the subspace $\mathcal{C}_2 \times \mathcal{C}_2$ of \mathbb{F}_4 -flows. Evaluations of the Tutte polynomial at the point $(-2, -\frac{1}{3})$ on H_4 as well as $(-3, 0)$, $(0, -3)$ and $(-1, -1)$ are also given an interpretation in Section 2 in terms of eulerian subgraphs of G .

By MacWilliams duality (see Section 1.3 below), the interpretations that we give for evaluations of the Tutte polynomial at points (a, b) in terms of eulerian subgraphs (cycles, \mathcal{C}_2) become interpretations for points (b, a) in terms of cutsets (cocycles, \mathcal{C}_2^\perp). In the same way, for exam-

ple, there is the expansion of the Ising model of a graph over its cutsets (bipartitions) [20, §4.3] corresponding to van der Waerden's expansion over eulerian subgraphs.

1.2. The Fourier transform

In this section we summarise the facts about the Fourier transform on rings such as \mathbb{F}_q and \mathbb{Z}_q (the integers modulo q) that the reader needs to be aware of in this paper.

Let Q be a commutative ring (either \mathbb{F}_q or \mathbb{Z}_q in the sequel) and Q^E the set of all vectors $x = (x_e : e \in E)$ with entries in Q indexed by E . The indicator function 1_k for $k \in Q$ is defined by $1_k(\ell) = 1$ if $k = \ell$ and 0 otherwise. A subset $S \subseteq Q$ has indicator function defined by $1_S = \sum_{k \in S} 1_k$.

A *character* of Q is a homomorphism $\chi : Q \rightarrow \mathbb{C}^\times$ from the additive group Q to the multiplicative group of \mathbb{C} . The set of characters form a group \widehat{Q} under pointwise multiplication isomorphic to Q as an abelian group. For Q^E the group of characters $\widehat{Q^E}$ is isomorphic to \widehat{Q}^E .

For each $k \in Q$, write χ_k for the image of k under a fixed isomorphism of Q with \widehat{Q} . In particular, the principal (trivial) character χ_0 is defined by $\chi_0(\ell) = 1$ for all $\ell \in Q$, and $\chi_{-k}(\ell) = \overline{\chi_k(\ell)}$ for all $k, \ell \in Q$, where the bar denotes complex conjugation. A character $\chi \in \widehat{Q}$ is a *generating character* for Q if $\chi_k(\ell) = \chi(k\ell)$ for each character $\chi_k \in \widehat{Q}$. The ring \mathbb{Z}_q has a generating character χ defined by $\chi(k) = e^{2\pi i k/q}$. (The fixed isomorphism $k \mapsto \chi_k$ of \mathbb{Z}_q with $\widehat{\mathbb{Z}_q}$ in this case is given by taking $\chi_k(\ell) = e^{2\pi i k\ell/q}$.) The field \mathbb{F}_q with $q = p^m$ has a generating character χ defined by $\chi(k) = e^{2\pi i \text{Tr}(k)/p}$, where $\text{Tr}(k) = k + k^p + \dots + k^{p^{m-1}}$ is the trace of k . (Here the isomorphism $\mathbb{F}_q \rightarrow \widehat{\mathbb{F}_q}$, $k \mapsto \chi_k$ is given by taking $\chi_k(\ell) = e^{2\pi i \text{Tr}(k\ell)/p}$.) If χ is a generating character for Q then $\chi^{\otimes E}$, defined by $\chi^{\otimes E}(x) = \prod_{e \in E} \chi(x_e)$ for $x = (x_e : e \in E) \in Q^E$, is a generating character for Q^E . The euclidean inner product (dot product) on Q^E is defined by $x \cdot y = \sum x_e y_e$. Since $\prod \chi(x_e) = \chi(\sum x_e)$, it follows that $\chi^{\otimes E}$ satisfies $\chi_x^{\otimes E}(y) = \chi(x \cdot y)$ for $x, y \in Q^E$.

The vector space \mathbb{C}^{Q^E} over \mathbb{C} of all functions from Q^E to \mathbb{C} is an inner product space with Hermitian inner product $\langle \cdot, \cdot \rangle$ defined for $f, g : Q^E \rightarrow \mathbb{C}$ by

$$\langle f, g \rangle = \sum_{x \in Q^E} f(x) \overline{g(x)}.$$

Fix an isomorphism $k \mapsto \chi_k$ of Q with \widehat{Q} and let χ be a generating character for Q such that $\chi_k(\ell) = \chi(k\ell)$. For $f \in \mathbb{C}^Q$ the *Fourier transform* $\hat{f} \in \mathbb{C}^Q$ is defined for $k \in Q$ by

$$\hat{f}(k) = \langle f, \chi_k \rangle = \sum_{\ell \in Q} f(\ell) \chi(-k\ell).$$

The Fourier transform of a function $g : Q^E \rightarrow \mathbb{C}$ is then given by

$$\hat{g}(x) = \sum_{y \in Q^E} g(y) \chi(-x \cdot y).$$

In the space \mathbb{C}^{Q^E} , the Fourier inversion formula is $\hat{\hat{f}}(x) = q^{|E|} f(-x)$, or

$$f(y) = q^{-|E|} \langle \hat{f}, \chi_{-y} \rangle = q^{-|E|} \sum_{x \in Q^E} \hat{f}(x) \chi(x \cdot y).$$

Plancherel's or Parseval's identity is $\langle f, g \rangle = q^{-|E|} \langle \hat{f}, \hat{g} \rangle$.

For a subset \mathcal{S} of Q^E , the annihilator $\mathcal{S}^\#$ of \mathcal{S} is defined by $\mathcal{S}^\# = \{y \in Q^E : \forall_{x \in \mathcal{S}} \chi_x(y) = 1\}$. The annihilator $\mathcal{S}^\#$ is a subgroup of Q^E isomorphic to Q^E/\mathcal{S} . When \mathcal{S} is a Q -submodule of Q^E and Q has a generating character, the annihilator of \mathcal{S} is equal to the orthogonal \mathcal{S}^\perp to \mathcal{S} (with respect to the euclidean inner product), defined by $\mathcal{S}^\perp = \{y \in Q^E : \forall_{x \in \mathcal{S}} x \cdot y = 0\}$.

A key property of the Fourier transform is that for a Q -submodule \mathcal{S} of Q^E

$$\widehat{1_{\mathcal{S}}}(y) = \sum_{x \in \mathcal{S}} \chi_x(y) = |\mathcal{S}| 1_{\mathcal{S}^\#}(y),$$

and the Poisson summation formula is that

$$\sum_{x \in \mathcal{S}} f(x+z) = \frac{1}{|\mathcal{S}^\#|} \sum_{x \in \mathcal{S}^\#} \widehat{f}(x) \chi_z(x).$$

For Q -submodule \mathcal{S} of Q^E the coset $\{x+z : x \in \mathcal{S}\}$ of \mathcal{S} in the additive group Q^E is denoted by $\mathcal{S}+z$, an element of the quotient module Q^E/\mathcal{S} .

1.3. Flows, tensions and Hamming weight enumerators

For the moment we continue with Q a commutative ring on q elements with a generating character. Let \mathcal{S} be a subset of Q^E , the set of vectors with entries in Q indexed by edges of G . The Hamming weight of a vector $x = (x_e : e \in E) \in Q^E$ is defined by $|x| = \#\{e \in E : x_e \neq 0\}$. The Hamming weight enumerator of \mathcal{S} is defined by

$$\text{hwe}(\mathcal{S}; t) = \sum_{x \in \mathcal{S}} t^{|E|-|x|},$$

the exponent being the number of zero entries in x .

When \mathcal{S} is a Q -submodule of Q^E the MacWilliams duality theorem states that

$$\text{hwe}(\mathcal{S}; t) = \frac{(t-1)^{|E|}}{|\mathcal{S}^\perp|} \text{hwe}\left(\mathcal{S}^\perp; \frac{t+q-1}{t-1}\right), \quad (1)$$

which follows from the Poisson summation formula with $f = (t1_0 + 1_{Q \setminus 0})^{\otimes E}$.

A Q -flow of G is defined with reference to a ground orientation γ of G ; the number of Q -flows of a given Hamming weight is independent of γ . A vector $x \in Q^E$ is a Q -flow of G if, for each vertex $v \in V$,

$$\sum_{e \in E} \gamma_{v,e} x_e = 0,$$

where $\gamma_{v,e} = +1$ if e is directed into v , $\gamma_{v,e} = -1$ if e is directed out of v , and $\gamma_{v,e} = 0$ if e is not incident with v . The Q -flows form a Q -submodule of Q^E whose orthogonal Q -submodule is the set of Q -tensions of G . The latter comprise the set of $y \in Q^E$ such that there exists a vertex Q -colouring $z \in Q^V$ with $y_e = z_u - z_v$ for all edges $e = (u, v)$ (directed by the orientation γ). To each Q -tension y there correspond $q^{k(G)}$ vertex Q -colourings for which $y_e = z_u - z_v$.

A nowhere-zero Q -flow has Hamming weight $|E|$, and likewise a nowhere-zero Q -tension. To a nowhere-zero Q -tension y corresponds a set of $q^{k(G)}$ proper vertex Q -colourings of G , i.e., $z \in Q^V$ such that $z_u \neq z_v$ whenever u is adjacent to v in G .

The Tutte polynomial on the hyperbola H_q is related to the Hamming weight enumerator of the set of Q -flows via the identity

$$\text{hwe}(Q\text{-flows of } G; t) = (t-1)^{n(G)} T\left(G; t, \frac{t+q-1}{t-1}\right). \quad (2)$$

By (1) the set of Q -tensions has Hamming weight enumerator

$$\text{hwe}(Q\text{-tensions of } G; t) = (t-1)^{r(G)} T\left(G; \frac{t+q-1}{t-1}, t\right).$$

2. The Tutte polynomial on H_2 and H_4

From Eq. (2), the Tutte polynomial specialises on H_2 to the Hamming weight enumerator of the space \mathcal{C}_2 of \mathbb{F}_2 -flows,

$$\text{hwe}(\mathcal{C}_2; t) = (t-1)^{n(G)} T\left(G; t, \frac{t+1}{t-1}\right).$$

If G is eulerian then the all 1 vector belongs to \mathcal{C}_2 with the consequence that $\text{hwe}(\mathcal{C}_2; t) = t^{|E|} \text{hwe}(\mathcal{C}_2; t^{-1})$, whence if G is eulerian then

$$T\left(G; t, \frac{t+1}{t-1}\right) = (-1)^{n(G)} t^{r(G)} T\left(G; \frac{1}{t}, \frac{1+t}{1-t}\right). \quad (3)$$

Dually, if G is bipartite then the all 1 vector belongs to \mathcal{C}_2^\perp and in this case

$$T\left(G; \frac{t+1}{t-1}, t\right) = (-1)^{r(G)} t^{n(G)} T\left(G; \frac{1+t}{1-t}, \frac{1}{t}\right).$$

Using the MacWilliams duality theorem (1),

$$\begin{aligned} (t-1)^{n(G)} T\left(G; t, \frac{t+1}{t-1}\right) \\ = \sum_{\text{eulerian } A \subseteq E} t^{|E|-|A|} = 2^{-r(G)} (t-1)^{|E|} \sum_{\text{cutsets } A \subseteq E} \left(\frac{t+1}{t-1}\right)^{|E|-|A|}. \end{aligned}$$

In particular

$$\sum_{\text{eulerian } A \subseteq E} (-1)^{|A|} = 2^{|E|-|V|} P(G; 2).$$

Likewise, the Tutte polynomial on H_4 is the Hamming weight enumerator of the space $\mathcal{C}_4 \cong \mathcal{C}_2 \times \mathcal{C}_2$ of \mathbb{F}_4 -flows,

$$\text{hwe}(\mathcal{C}_2 \times \mathcal{C}_2; t) = (t-1)^{n(G)} T\left(G; t, \frac{t+3}{t-1}\right).$$

The MacWilliams duality theorem (1) here is that

$$\text{hwe}(\mathcal{C}_2 \times \mathcal{C}_2; t) = \frac{(t-1)^{|E|}}{|\mathcal{C}_2^\perp|^2} \text{hwe}\left(\mathcal{C}_2^\perp \times \mathcal{C}_2^\perp; \frac{t+3}{t-1}\right),$$

which in terms of eulerian subgraphs of G says that

$$\sum_{\text{eulerian } A, B \subseteq E} t^{|E|-|A \cup B|} = 4^{-r(G)} (t-1)^{|E|} \sum_{\text{cutsets } A, B \subseteq E} \left(\frac{t+3}{t-1}\right)^{|E|-|A \cup B|}.$$

In particular

$$\sum_{\text{eulerian } A, B \subseteq E} (-3)^{|E|-|A \cup B|} = (-1)^{|E|} 4^{|E|-|V|} P(G; 4).$$

2.1. Coset weight enumerators

In a previous article [7] the following specialisations of the Tutte polynomial to the hyperbola H_2 and the hyperbola H_4 were derived as an illustration of the techniques afforded by elementary Fourier analysis. See also [11] for these identities in the context of coding theory. In this section we give a combinatorial interpretation of these identities for particular values of t and derive evaluations of the Tutte polynomial on H_2 and H_4 .

When writing $C_2 + z \in \mathbb{F}_2^E / C_2$ in the range of summations below, we assume that $z \in \mathbb{F}_2^E$ ranges over a transversal of the cosets, each coset $C_2 + z$ appearing only once.

Lemma 1. *Let $G = (V, E)$ be a graph and let C_2 be the subspace of \mathbb{F}_2 -flows of G . Then, for $t \in \mathbb{C}$,*

$$\sum_{C_2+z \in \mathbb{F}_2^E / C_2} |\text{hwe}(C_2 + z; t)|^2 = (t + \bar{t})^{r(G)} |t - 1|^{2n(G)} T\left(G; \frac{|t|^2 + 1}{t + \bar{t}}, \left|\frac{t + 1}{t - 1}\right|^2\right),$$

$$\sum_{C_2+z \in \mathbb{F}_2^E / C_2} \text{hwe}(C_2 + z; t)^2 = (2t)^{r(G)} (t - 1)^{2n(G)} T\left(G; \frac{t^2 + 1}{2t}, \left(\frac{t + 1}{t - 1}\right)^2\right),$$

and

$$\sum_{C_2+z \in \mathbb{F}_2^E / C_2} \text{hwe}(C_2 + z; t)^3 = (t + 1)^{|E|} t^{r(G)} (t - 1)^{2n(G)} T\left(G; \frac{t^2 - t + 1}{t}, \left(\frac{t + 1}{t - 1}\right)^2\right).$$

Note that $\left(\frac{t+1}{t-1}\right)^2$ is real if and only if $t \in \mathbb{R}$ or $|t| = 1$, since $\frac{t+1}{t-1} = \frac{\bar{t}-t+|t|^2-1}{|t|^2+1-t-\bar{t}}$. By putting $t = e^{i\theta}$ in the identities of Lemma 1, routine calculations yield the following.¹

Corollary 2. *If $t = e^{i\theta}$ for some $\theta \in (0, 2\pi)$ then*

$$\begin{aligned} 2^{-|E|} \sum_{C_2+z \in \mathbb{F}_2^E / C_2} |\text{hwe}(C_2 + z; e^{i\theta})|^2 \\ = (\cos \theta)^{r(G)} (1 - \cos \theta)^{n(G)} T\left(G; \frac{1}{\cos \theta}, \frac{1 + \cos \theta}{1 - \cos \theta}\right), \end{aligned} \quad (4)$$

$$(2e^{i\theta})^{-|E|} \sum_{C_2+z \in \mathbb{F}_2^E / C_2} \text{hwe}(C_2 + z; e^{i\theta})^2 = (\cos \theta - 1)^{n(G)} T\left(G; \cos \theta, \frac{\cos \theta + 1}{\cos \theta - 1}\right), \quad (5)$$

¹ From [7, Corollary 7.5] it is readily seen that for $r \in \mathbb{N}$ the sum of r th powers of the coset weight enumerators of C_2 is a specialisation of the *complete weight enumerator* (see Section 3 for a definition) of the $(r - 1)$ -fold direct product $C_2^\perp \times \cdots \times C_2^\perp$ (isomorphic to the space of $\mathbb{F}_{2^{r-1}}$ -tensions of G). Only for $r \leq 3$ is this specialisation a Hamming weight enumerator, and so the sum of r th powers of coset Hamming weight enumerators of C_2 is a specialisation of the Tutte polynomial only for $r \leq 3$. For example, the sum of fourth powers of the coset weight enumerators $\text{hwe}(C_2 + z; t)$ turns out to be equal to

$$2^{-3r(G)} (t^2 - 1)^{2|E|} \sum_{\text{eulerian } A, B, C \subseteq E} s^{|E| - |A \cup B \cup C| - |A \cap B \cap C|}$$

where $s = \left(\frac{t+1}{t-1}\right)^2$.

Table 1
Evaluations of the Tutte polynomial in the identities of Corollary 2

θ	$\cos \theta$	Eq. (4), point on H_2	Eq. (5), point on H_2	Eq. (6), point on H_4
π	-1	$(-1, 0)$	$(-1, 0)$	$(-3, 0)$
$2\pi/3$	$-\frac{1}{2}$	$(-2, \frac{1}{3})$	$(-\frac{1}{2}, -\frac{1}{3})$	$(-2, -\frac{1}{3})$
$\pi/2$	0	$*$	$(0, -1)$	$(-1, -1)$
$\pi/3$	$\frac{1}{2}$	$(2, 3)$	$(\frac{1}{2}, -3)$	$(0, -3)$

* For $\theta = \pi/2$, the right-hand side of Eq. (4) is equal to 1 independent of G .

For $\theta = \pi$ the factor $(1 + \cos \theta)^{\frac{1}{2}|E|}$ on the right-hand side of (6) is equal to zero.

and

$$\begin{aligned}
 & (2e^{i\theta})^{-\frac{3}{2}|E|} \sum_{C_2 + z \in \mathbb{R}_2^E / C_2} \text{hwe}(C_2 + z; e^{i\theta})^3 \\
 &= 2^{-r(G)} (1 + \cos \theta)^{\frac{1}{2}|E|} (\cos \theta - 1)^{n(G)} T\left(G; 2\cos \theta - 1, \frac{\cos \theta + 1}{\cos \theta - 1}\right). \quad (6)
 \end{aligned}$$

If $\theta/2\pi \in \mathbb{Q}$ then $2\cos \theta = e^{i\theta} + e^{-i\theta}$ is an algebraic integer, and hence an ordinary integer if $\cos \theta \in \mathbb{Q}$. The only rational values of $\cos \theta$ for $\theta \in \mathbb{Q}$ are thus $0, \pm\frac{1}{2}, \pm 1$, corresponding to $\theta = \pm\frac{\pi}{2}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pi, 0$. Thus when $e^{i\theta}$ is a q th root of unity for $q \in \{2, 3, 4, 6\}$ the evaluations of the Tutte polynomial in Corollary 2 are at rational points (see Table 1).

In order to give combinatorial interpretations of identity (4) (and identity (5)) we shall be interested in the correlation between two types of event when choosing $A, B \subseteq E$ uniformly at random. First, the event that $A \triangle B$ is eulerian. Second, for various integers q , the event that $|A| - |B|$ (respectively $|A| + |B|$) belongs to a certain subset of congruence classes modulo q .

Similarly, in order to interpret identity (6) we choose $A, B, C \subseteq E$ uniformly at random and look at the correlation between the event that $A \triangle B, B \triangle C, C \triangle A$ are each eulerian and the event that $|A| + |B| + |C|$ belongs to a certain subset of congruence classes modulo q .

2.2. Bias

If q is not a power of two, for fixed $k \in \{0, 1, \dots, q-1\}$ none of the events $|A| - |B| \equiv k \pmod{q}$, $|A| + |B| \equiv k \pmod{q}$ or $|A| + |B| + |C| \equiv k \pmod{q}$ can have probability $\frac{1}{q}$ when $A, B, C \subseteq E$ are taken uniformly at random. As observed in Remark 4 below, amongst powers of two only for $q = 2$ is it true that the values of $|A| \pm |B| (+|C|)$ are equidistributed modulo q (although it remains possible that for some values of k the event $|A| \pm |B| (+|C|) \equiv k \pmod{q}$ has probability $\frac{1}{q}$).

Let Σ be an event in the uniform probability space on pairs $A, B \subseteq E$ or triples $A, B, C \subseteq E$, and $\overline{\Sigma}$ is its complement. In the sequel the event Σ takes the form $|A| \pm |B| \in S \pmod{q}$ or $|A| + |B| + |C| \in S \pmod{q}$ for a subset S of the integers $\{0, 1, \dots, q-1\}$ modulo q .

Define the *bias* towards Σ by

$$\text{Bias}(\Sigma) = \mathbb{P}(\Sigma) - \mathbb{P}(\overline{\Sigma}).$$

Note that $\text{Bias}(\overline{\Sigma}) = -\text{Bias}(\Sigma)$ and the event Σ has probability $\frac{1}{2}[1 + \text{Bias}(\Sigma)]$.

For example, if $q = 2$ and $S = \{0\}$ then $\Sigma = \{A, B \subseteq E: |A| + |B| \in S \pmod{q}\}$ is the event that $|A| + |B|$ is even and this has the same probability as the event $\overline{\Sigma} = \{A, B \subseteq E: |A| + |B| \notin S \pmod{q}\}$.

$S \pmod{q}$ that $|A| + |B|$ is odd. For $q > 2$, the bias of an event of the form $|A| \pm |B| \in S \pmod{q}$ or $|A| + |B| + |C| \in S \pmod{q}$ is usually not equal to zero. This is excepting the case when q is even and $S = \{0, 2, \dots, q-2\}$ or $S = \{1, 3, \dots, q-1\}$ for which the event Σ is about the parity of $|A| \pm |B| (+|C|)$ again. However, in Theorem 19 it is found that $\text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4}) = 0$ when $|E| \equiv 1 \pmod{4}$.

Lemma 3. Suppose $A, B, C \subseteq E$ are chosen uniformly at random and that $S \subseteq \{0, 1, \dots, q-1\}$. Then

$$\text{Bias}(\Sigma) = 2q^{-1} \langle \hat{g}, \hat{1}_S \rangle - 1,$$

where 1_S is the indicator function of S and

$$\hat{g}(k) = \begin{cases} 2^{-|E|} (1 + \cos \frac{2\pi k}{q})^{|E|}, \\ 2^{-|E|} e^{-2\pi i k |E|/q} (1 + \cos \frac{2\pi k}{q})^{|E|}, \\ 2^{-\frac{3}{2}|E|} e^{-2\pi i k \frac{3}{2}|E|/q} (1 + \cos \frac{2\pi k}{q})^{\frac{3}{2}|E|}, \end{cases}$$

according to

$$\Sigma = \begin{cases} \{A, B \subseteq E: |A| - |B| \in S \pmod{q}\}, \\ \{A, B \subseteq E: |A| + |B| \in S \pmod{q}\}, \\ \{A, B, C \subseteq E: |A| + |B| + |C| \in S \pmod{q}\}. \end{cases}$$

Proof. We prove the lemma for the event $\Sigma = \{A, B \subseteq E: |A| - |B| \in S \pmod{q}\}$. The other cases are similar.

Define

$$g(\ell) = \mathbb{P}(|A| - |B| \equiv \ell \pmod{q}).$$

Then

$$\mathbb{P}(\Sigma) = \sum_{\ell \in S} g(\ell) = \langle g, 1_S \rangle = q^{-1} \langle \hat{g}, \hat{1}_S \rangle,$$

using Plancherel's formula at the end. By definition, $\text{Bias}(\Sigma) = 2\mathbb{P}(\Sigma) - 1$, and the first part of the lemma is proved. It remains to calculate $\hat{g}(k)$ for $k \in \{0, 1, \dots, q-1\}$:

$$\begin{aligned} \hat{g}(k) &= \sum_{0 \leq \ell \leq q-1} e^{-2\pi i k \ell / q} g(\ell) = 2^{-2|E|} \sum_{A, B \subseteq E} e^{-2\pi i k (|A| - |B|) / q} \\ &= 2^{-2|E|} \sum_{A \subseteq E} e^{-2\pi i k |A| / q} \sum_{B \subseteq E} e^{2\pi i k |B| / q} \\ &= 2^{-2|E|} (1 + e^{-2\pi i k / q})^{|E|} (1 + e^{2\pi i k / q})^{|E|} \\ &= 2^{-2|E|} (e^{2\pi i k / q} + 2 + e^{-2\pi i k / q})^{|E|} \\ &= 2^{-|E|} \left(1 + \cos \frac{2\pi k}{q} \right)^{|E|}. \end{aligned}$$

This completes the proof. \square

Remark 4. In the notation of the proof of Lemma 3, the probabilities $g(\ell) = \mathbb{P}(|A| - |B| = \ell \pmod{q})$ are equal for all $\ell \in \{0, 1, \dots, q-1\}$ if and only if

$$\sum_{0 \leq \ell \leq q-1} g(\ell) e^{-2\pi i \ell / q} = 0.$$

But the left-hand sum is equal to $\hat{g}(1) = 2^{-|E|} (1 + \cos \frac{2\pi}{q})^{|E|}$, and this is equal to zero if and only if $q = 2$. Hence the values $|A| - |B|$ for $A, B \subseteq E$ are equidistributed modulo q if and only if $q = 2$. Similarly, the values $|A| + |B|$ for $A, B \subseteq E$ and the values of $|A| + |B| + |C|$ for $A, B, C \subseteq E$ are only equidistributed modulo q when $q = 2$.

Let Δ be an event in the uniform probability space on pairs $A, B \subseteq E$ or triples $A, B, C \subseteq E$. We define the conditional bias of Σ given Δ by

$$\text{Bias}(\Sigma \mid \Delta) = \mathbb{P}(\Sigma \mid \Delta) - \mathbb{P}(\overline{\Sigma} \mid \Delta).$$

In what follows, Δ is either the event that $A \triangle B$ is eulerian (where Σ is one of the events $|A| \pm |B| \in S \pmod{q}$ for some $S \subseteq \{0, 1, \dots, q-1\}$) or the event that $A \triangle B$ and $B \triangle C$ are both eulerian (where Σ is the event that $|A| + |B| + |C| \in S \pmod{q}$ for some $S \subseteq \{0, 1, \dots, q-1\}$). The covariance of (the indicator functions of) the events Σ and Δ is defined by the difference $\mathbb{P}(\Sigma \cap \Delta) - \mathbb{P}(\Sigma)\mathbb{P}(\Delta)$. We define the *correlation*² between the events Σ and Δ by dividing the covariance through by $\mathbb{P}(\Delta)$,

$$\text{Correlation}(\Sigma \mid \Delta) = \mathbb{P}(\Sigma \mid \Delta) - \mathbb{P}(\Sigma).$$

Correlation is related to bias via the relation

$$\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma) = 2 \text{Correlation}(\Sigma \mid \Delta).$$

When $\text{Bias}(\Sigma) \neq 0$ we shall have occasion to also measure correlation via the ratio

$$\frac{\text{Bias}(\Sigma \mid \Delta)}{\text{Bias}(\Sigma)} = \frac{2 \text{Correlation}(\Sigma \mid \Delta)}{\text{Bias}(\Sigma)} + 1. \quad (7)$$

This can be viewed as the scale factor from the existing bias towards Σ to the bias towards Σ given the event Δ . When the ratio (7) is greater than 1 the existing bias towards Σ is magnified, when the ratio is less than 1 the existing bias is diminished. The ratio (7) is equal to 1 if and only if there is no correlation between the events Σ and Δ .

In the next section we derive general expressions for $\text{Bias}(\Sigma)$ and $\text{Bias}(\Sigma \mid \Delta)$, for any choice of q and S in the definition of Σ , the latter expressed in terms of the Tutte polynomial evaluations of Corollary 2 for $\theta \in \{2\pi k/q: k = 1, \dots, q-1\}$. By taking $q \in \{2, 3, 4, 6\}$ we obtain interpretations for various evaluations of the Tutte polynomial at the points listed in Table 1. The reason why we limit ourselves to $q \in \{2, 3, 4, 6\}$ is not specifically on account of these corresponding to evaluations at rational points, but rather that only for these values of q do we obtain evaluations of the Tutte polynomial at a single point rather than a sum of evaluations at two or more distinct points.

² The *correlation coefficient* of the indicator functions of the events Σ and Δ is another normalisation of the covariance, namely

$$\frac{\mathbb{P}(\Sigma \cap \Delta) - \mathbb{P}(\Sigma)\mathbb{P}(\Delta)}{\sqrt{(\mathbb{P}(\Sigma)\mathbb{P}(\Delta)(1 - \mathbb{P}(\Sigma))(1 - \mathbb{P}(\Delta)))}}.$$

An evaluation of the Tutte polynomial at a single point is of interest as such an evaluation is a *Tutte–Grothendieck invariant* on graphs, i.e., satisfies a “linear” deletion–contraction recurrence relation. Specifically, if \setminus denotes deletion and $/$ contraction, a function f on graphs satisfying

$$f(G) = \begin{cases} af(G \setminus e) + bf(G/e) & \text{if } e \text{ is neither an isthmus nor a loop,} \\ xf(G/e) & \text{if } e \text{ is an isthmus,} \\ yf(G \setminus e) & \text{if } e \text{ is a loop,} \end{cases}$$

and with value $c^{|V|}$ on the edgeless graph (V, \emptyset) is given by the evaluation³ [4,20],

$$f(G) = c^{k(G)} a^{n(G)} b^{r(G)} T(G; cx/b, y/a).$$

Many of the evaluations of the Tutte polynomial in Sections 2.3.1 to 2.3.4 have been highlighted as theorems either because they have other well-known combinatorial meanings (such as the number of nowhere-zero 4-flows in Theorem 20) and the opaqueness of the connection to these other interpretations is intriguing, or because they on the contrary do not have such other well-known interpretations (such as $T(G; -2, -\frac{1}{3})$ in Theorem 16). Direct proofs of the corresponding deletion–contraction recurrences do not seem straightforward in many cases.

2.3. Evaluations of coset weight enumerators at q th roots of unity

In this section the identities in Corollary 2 are given interpretations in terms of $\text{Bias}(\Sigma \mid \Delta)$ where Σ takes the form $|A| \pm |B| (+|C|) \in S \pmod{q}$ and Δ is the event that $A \triangle B$ (and $B \triangle C$) is eulerian. These interpretations in their general form make for rather dull reading, but their particular cases for $q \in \{2, 3, 4, 6\}$ are the more interesting theorems that follow as corollaries. The latter are presented in Sections 2.3.1 to 2.3.4 to which the reader might wish to turn before referring back to this section.

Lemma 5. Suppose $A, B \subseteq E$ are chosen uniformly at random and Δ is the event that $A \triangle B$ is eulerian. Then, for any $k \in \{0, 1, \dots, q-1\}$,

$$\begin{aligned} & 2^{-|E|} \sum_{\mathcal{C}_2 + z \in \mathbb{F}_2^E / \mathcal{C}_2} |\text{hwe}(\mathcal{C}_2 + z; e^{2\pi i k / q})|^2 \\ &= 2^{n(G)} \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell / q} \mathbb{P}(|B| - |A| \equiv \ell \pmod{q} \mid \Delta). \end{aligned}$$

Proof. Given a coset $\mathcal{C}_2 + z$ and $\ell \in \{0, \dots, q-1\}$, define

$$p_\ell = p_\ell(\mathcal{C}_2 + z) = \#\{x \in \mathcal{C}_2 + z: |E| - |x| \equiv \ell \pmod{q}\},$$

³ An *isthmus* or bridge is an edge forming a cutset of size 1 and a *loop* is an edge forming a cycle of size 1. If $b = 0$ then

$$f(G) = c^{|V|} a^{n(G) - \ell(G)} x^{r(G)} y^{\ell(G)},$$

and if $a = 0$ then

$$f(G) = c^{k(G) + i(G)} b^{r(G) - i(G)} x^{i(G)} y^{n(G)},$$

where $i(G)$ is the number of isthmuses and $\ell(G)$ the number of loops in G .

so that

$$\sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell / q} p_\ell = \text{hwe}(\mathcal{C}_2 + z; e^{2\pi i k / q}).$$

Going on to define

$$P_\ell = P_\ell(\mathcal{C}_2 + z) = \sum_{j-k \equiv \ell \pmod{q}} p_j p_k,$$

we have

$$|\text{hwe}(\mathcal{C}_2 + z; e^{2\pi i k / q})|^2 = \left| \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell / q} p_\ell \right|^2 = \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell / q} P_\ell. \quad (8)$$

Let $C \subseteq E$ have indicator vector $z \in \mathbb{F}_2^E$ and suppose $A \subseteq E$ is chosen uniformly at random with indicator vector $x \in \mathbb{F}_2^E$. Then $|E| - |x| = |E \setminus A|$ and $x \in \mathcal{C}_2 + z$ if and only if $x + z \in \mathcal{C}_2$, i.e., $A \triangle C$ is eulerian. Hence

$$2^{-|E|} p_\ell(\mathcal{C}_2 + z) = \mathbb{P}(|E \setminus A| \equiv \ell \pmod{q} \cap A \triangle C \text{ eulerian}).$$

Similarly, if $A, B \subseteq E$ are chosen uniformly at random then

$$\begin{aligned} 2^{-2|E|} P_\ell(\mathcal{C}_2 + z) &= \sum_{j-k \equiv \ell \pmod{q}} \mathbb{P}(|E \setminus A| \equiv j \pmod{q} \cap A \triangle C \text{ eulerian}) \\ &\quad \times \mathbb{P}(|E \setminus B| \equiv k \pmod{q} \cap B \triangle C \text{ eulerian}) \\ &= \mathbb{P}(|E \setminus A| - |E \setminus B| \equiv \ell \pmod{q} \cap A \triangle C \text{ eulerian} \cap B \triangle C \text{ eulerian}). \end{aligned}$$

Given $C, C' \subseteq E$, the events $\{A \triangle C \text{ eulerian} \cap B \triangle C \text{ eulerian}\}$ and $\{A \triangle C' \text{ eulerian} \cap B \triangle C' \text{ eulerian}\}$ are either equal (when $C \triangle C'$ is eulerian) or disjoint. For suppose that C has indicator vector z and C' indicator vector z' . If $x + z \in \mathcal{C}_2$ and $x + z' \in \mathcal{C}_2$ then $z + z' \in \mathcal{C}_2$, i.e., $C \triangle C'$ is eulerian. Conversely, if $z + z' \in \mathcal{C}_2$ and $x + z \in \mathcal{C}_2$ then $x + z + (z + z') = x + z' \in \mathcal{C}_2$.

Letting $C \subseteq E$ range over a collection of subsets no two of which have eulerian symmetric difference, the union of events $\{A \triangle C \text{ eulerian} \cap B \triangle C \text{ eulerian}\}$ is thus a disjoint union and equal to the event $\{A \triangle B \text{ eulerian}\} = \Delta$. Hence, letting z range over a transversal of cosets of \mathcal{C}_2 ,

$$2^{-2|E|} \sum_{\mathcal{C}_2 + z \in \mathbb{F}_2^E / \mathcal{C}_2} P_\ell(\mathcal{C}_2 + z) = \mathbb{P}(|E \setminus A| - |E \setminus B| \equiv \ell \pmod{q} \cap \Delta).$$

From Eq. (8) it follows that

$$\begin{aligned} 2^{-2|E|} \sum_{\mathcal{C}_2 + z \in \mathbb{F}_2^E / \mathcal{C}_2} |\text{hwe}(\mathcal{C}_2 + z; e^{2\pi i k / q})|^2 &= \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell / q} \mathbb{P}(|B| - |A| \equiv \ell \pmod{q} \cap \Delta). \end{aligned}$$

Dividing through by $\mathbb{P}(\Delta) = 2^{-r(G)}$ gives the result. \square

Lemma 6. Suppose that $A, B \subseteq E$ are chosen uniformly at random and Δ is the event that $A \triangle B$ is eulerian. Then, for any $k \in \{0, 1, \dots, q-1\}$,

$$\begin{aligned} 2^{-|E|} \sum_{C_2+z \in \mathbb{F}_2^E/C_2} \text{hwe}(C_2+z; e^{2\pi i k/q})^2 \\ = 2^{n(G)} \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell/q} \mathbb{P}(|A| + |B| \equiv \ell \pmod{q} \mid \Delta). \end{aligned}$$

Proof. The same mutatis mutandis as the proof of Lemma 5 with

$$P_\ell = P_\ell(C_2+z) = \sum_{j+k \equiv \ell \pmod{q}} p_j p_k$$

replacing the definition of P_ℓ given in the proof of that lemma. Note that since $A, B \subseteq E$ are chosen uniformly at random and $(E \setminus A) \triangle (E \setminus B) = A \triangle B$, by symmetry we have $\mathbb{P}(|E \setminus A| + |E \setminus B| \equiv \ell \pmod{q} \mid \Delta) = \mathbb{P}(|A| + |B| \equiv \ell \pmod{q} \mid \Delta)$. \square

Lemma 7. Suppose that $A, B, C \subseteq E$ are chosen uniformly at random and Δ is the event that $A \triangle B, B \triangle C$ are both eulerian. Then, for any $k \in \{0, 1, \dots, q-1\}$,

$$\begin{aligned} 2^{-|E|} \sum_{C_2+z \in \mathbb{F}_2^E/C_2} \text{hwe}(C_2+z; e^{2\pi i k/q})^3 \\ = 4^{n(G)} \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell/q} \mathbb{P}(|A| + |B| + |C| \equiv \ell \pmod{q} \mid \Delta). \end{aligned}$$

Proof. The same as the proof of Lemma 5 with the following being the main alterations. Set

$$P_\ell = P_\ell(C_2+z) = \sum_{i+j+k \equiv \ell \pmod{q}} p_i p_j p_k.$$

Then

$$\text{hwe}(C_2+z; e^{2\pi i k/q})^3 = \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell/q} P_\ell$$

and

$$2^{-3|E|} \sum_{C_2+z \in \mathbb{F}_2^E/C_2} P_\ell(C_2+z) = \mathbb{P}(|E \setminus A| + |E \setminus B| + |E \setminus C| \equiv \ell \pmod{q} \mid \Delta).$$

As in the proof of Lemma 6, we can by symmetry replace $E \setminus A, E \setminus B, E \setminus C$ by A, B, C . Hence

$$\begin{aligned} 2^{-3|E|} \sum_{C_2+z \in \mathbb{F}_2^E/C_2} \text{hwe}(C_2+z; e^{2\pi i k/q})^3 \\ = \sum_{0 \leq \ell \leq q-1} e^{2\pi i k \ell/q} \mathbb{P}(|A| + |B| + |C| \equiv \ell \pmod{q} \mid \Delta). \end{aligned}$$

Dividing through by $\mathbb{P}(\Delta) = 2^{-2r(G)}$ gives the result. \square

Lemma 8. Suppose that $A, B, C \subseteq E$ are chosen uniformly at random and Δ is the event that $A \triangle B$ (and $B \triangle C$) is eulerian. Then

$$\text{Bias}(\Sigma \mid \Delta) = 2q^{-1} \langle \hat{f}, \hat{1}_S \rangle - 1,$$

where 1_S is the indicator function of S and, for $k \in \{0, 1, \dots, q-1\}$,

$$\hat{f}(k) = \begin{cases} 2^{-n(G)} \left(\cos \frac{2\pi k}{q} \right)^{r(G)} \left(1 - \cos \frac{2\pi k}{q} \right)^{n(G)} T\left(G; \frac{1}{\cos \frac{2\pi k}{q}}, \frac{1 + \cos \frac{2\pi k}{q}}{1 - \cos \frac{2\pi k}{q}}\right), \\ 2^{-n(G)} e^{-2\pi i k |E|/q} \left(\cos \frac{2\pi k}{q} - 1 \right)^{n(G)} T\left(G; \cos \frac{2\pi k}{q}, \frac{\cos \frac{2\pi k}{q} + 1}{\cos \frac{2\pi k}{q} - 1}\right), \\ 2^{-n(G) - \frac{1}{2}|E|} e^{-2\pi i k \frac{3}{2}|E|/q} \left(1 + \cos \frac{2\pi k}{q} \right)^{\frac{1}{2}|E|} \left(\cos \frac{2\pi k}{q} - 1 \right)^{n(G)} \\ \quad \times T\left(G; 2 \cos \frac{2\pi k}{q} - 1, \frac{\cos \frac{2\pi k}{q} + 1}{\cos \frac{2\pi k}{q} - 1}\right) \end{cases}$$

according as

$$\Sigma = \begin{cases} \{A, B \subseteq E: |A| - |B| \in S \pmod{q}\}, \\ \{A, B \subseteq E: |A| + |B| \in S \pmod{q}\}, \\ \{A, B, C \subseteq E: |A| + |B| + |C| \in S \pmod{q}\}. \end{cases}$$

Proof. Again we prove the result for $\Sigma = \{A, B \subseteq E: |A| - |B| \in S \pmod{q}\}$ and Δ the event that $A \triangle B$ is eulerian, the other cases being entirely similar.

Define $f(\ell) = \mathbb{P}(|A| - |B| \equiv \ell \mid \Delta)$. Then $\mathbb{P}(\Sigma) = \langle f, 1_S \rangle = q^{-1} \langle \hat{f}, \hat{1}_S \rangle$, by Parseval's formula. By Lemma 5,

$$\hat{f}(k) = 2^{-|E| - n(G)} \sum_{C_2 + z \in \mathbb{F}_2^E / C_2} |\text{hwe}(C_2 + z; e^{-2\pi i k/q})|^2$$

and identity (4) gives the result. \square

Lemma 8 shows that $\text{Bias}(\Sigma \mid \Delta)$ is given in terms of evaluations of the Tutte polynomial at one or more points. The remainder of this section is spent establishing when an evaluation at just one point is involved and a Tutte–Grothendieck invariant results.

Note that $\hat{f}(0) = 1$ for each \hat{f} defined in Lemma 8 (f defines a probability distribution on \mathbb{Z}_q and $\hat{f}(0) = \sum f(k) = 1$). Recall the definition of g from Lemma 3, where $\text{Bias}(\Sigma)$ is expressed in terms of the inner product $\langle \hat{g}, \hat{1}_S \rangle$.

Since f and g are real-valued, $\hat{g}(-k) = \overline{\hat{g}(k)}$ and $\hat{f}(-k) = \overline{\hat{f}(k)}$. Remark also that if q is even and $\Sigma = \{A, B, C \subseteq E: |A| + |B| + |C| \in S \pmod{q}\}$ then $\hat{f}(q/2) = 0$ (for all graphs G), but that $\hat{f}(k) \neq 0$ for some graph G when $k \neq q/2$.

The support of a function $h: Q \rightarrow \mathbb{C}$ is defined by $\text{supp}(h) = \#\{k \in Q: h(k) \neq 0\}$. Thus $\text{supp}(\hat{g}) = \mathbb{Z}_q \setminus \{q/2\}$, and $\text{supp}(\hat{f}) = \mathbb{Z}_q$ (or possibly $\mathbb{Z}_q \setminus \{q/2\}$ as we have just seen). From Lemma 8, $\text{Bias}(\Sigma \mid \Delta) = 2q^{-1} \langle \hat{f}, \hat{1}_S \rangle - 1$ will involve an evaluation of the Tutte polynomial at a single point (valid for all graphs G) only if $\text{supp}(\hat{f} \cdot \hat{1}_S) \subseteq \{0, \ell, -\ell\}$ for some ℓ , or $\text{supp}(\hat{f} \cdot \hat{1}_S) \subseteq \{0, \ell, -\ell, q/2\}$ if q is even and $\Sigma = \{A, B, C \subseteq E: |A| + |B| + |C| \in S \pmod{q}\}$. This is so that the only non-zero terms contributing to the expression $2q^{-1} \langle \hat{f}, \hat{1}_S \rangle - 1$ are $\hat{f}(\ell) \overline{\hat{1}_S(\ell)}$ and its complex conjugate $\hat{f}(-\ell) \overline{\hat{1}_S(-\ell)}$.

Suppose Σ' is an event defined just as Σ except with $S' \subseteq \mathbb{Z}_q \setminus S$ replacing S . Then the bias towards S can be compared to the bias toward S' by considering the difference $\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma' \mid \Delta)$. When $S' = \mathbb{Z}_q \setminus S$ this difference is simply $2\text{Bias}(\Sigma \mid \Delta)$. The criterion for $\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma' \mid \Delta)$ to be an evaluation of the Tutte polynomial at a single point valid for all graphs is that $\text{supp}(\hat{f} \cdot \widehat{1_S - 1_{S'}}) \subseteq \{0, \ell, -\ell, q/2\}$ (with $q/2$ included under the same conditions as before).

The multiplicative group of units of \mathbb{Z}_q is denoted by \mathbb{Z}_q^\times and has order $\phi(q)$, where $\phi(q) = \#\{1 \leq k \leq q: (k, q) = 1\}$ is Euler's totient function. For $S \subseteq \mathbb{Z}_q$ and $\ell \in \mathbb{Z}_q$ we write $\ell S = \{\ell s: s \in S\}$.

Lemma 9. *If $h: \mathbb{Z}_q \rightarrow \mathbb{Q}$ and $\hat{h}(k) \neq 0$ then $\text{supp}(\hat{h}) \supseteq k\mathbb{Z}_q^\times$.*

If $\text{supp}(\hat{h}) \subseteq d\mathbb{Z}_q$ for some divisor d of q then h is constant on cosets of $(q/d)\mathbb{Z}_q$.

Thus if $\hat{h}(k) \neq 0$ for a unit k of \mathbb{Z}_q then $\hat{h}(\ell) \neq 0$ for all $\ell \in \mathbb{Z}_q^\times$, while if there is no unit k in the support of \hat{h} then h is constant on cosets of a proper subgroup of \mathbb{Z}_q .

Proof. The first statement depends on the fact that h takes rational values. Suppose $j \mapsto \sigma_j$ is the natural isomorphism of the group of Galois automorphisms of $\mathbb{Q}(e^{2\pi i/q})$ with \mathbb{Z}_q^\times , i.e., $\sigma_j: e^{2\pi i/q} \mapsto e^{2\pi i j/q}$. Then

$$\sigma_j(\hat{h}(k)) = \sigma_j\left(\sum_{0 \leq \ell \leq q-1} h(\ell) e^{2\pi i k \ell / q}\right) = \sum_{0 \leq \ell \leq q-1} h(\ell) \sigma_j(e^{2\pi i k \ell / q}) = \hat{h}(jk).$$

Hence $\hat{h}(k) \neq 0$ implies $\hat{h}(jk) \neq 0$ for all $j \in \mathbb{Z}_q^\times$.

For the second statement, the assumption is that $\hat{h} = \sum_{0 \leq \ell \leq q/d-1} \hat{h}(d\ell) 1_{d\ell}$. Taking Fourier transforms, for each $k \in \mathbb{Z}_q$ we have

$$qh(k) = \sum_{0 \leq \ell \leq q/d-1} \hat{h}(d\ell) e^{2\pi i k \ell / (q/d)}.$$

Then

$$h(k + q/d) = q^{-1} \sum_{0 \leq \ell \leq q/d-1} \hat{h}(d\ell) e^{2\pi i k \ell / (q/d) + 2\pi i \ell} = h(k),$$

so that h has period q/d and is constant on additive cosets of $(q/d)\mathbb{Z}_q$. \square

Whether $\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma' \mid \Delta)$ involves an evaluation of the Tutte polynomial at a single point depends on how many zero terms there are in its expression as $2q^{-1} \langle \hat{f}, \widehat{1_S - 1_{S'}} \rangle$ obtained from Lemma 8 applied to Σ and Σ' . We require $|S| = |S'|$ for the sets S, S' defining the events Σ, Σ' since $\widehat{1_S}(0) - \widehat{1_{S'}}(0) = |S| - |S'|$ and $\hat{f}(0) = 1$.

Corollary 10. *Suppose that Σ is one of the events $\{A, B \subseteq E: |A| \pm |B| \in S \pmod{q}\}$, Σ' is similarly defined with $S' \subseteq \mathbb{Z}_q \setminus S$ in place of S , and Δ is the event that $A \triangle B$ is eulerian. Then $\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma' \mid \Delta)$ is up to a factor depending only on $|E|$ and $r(G)$ an evaluation of the Tutte polynomial of G at a single point only if $|S| = |S'|$ and $q \in \{2, 3, 4\}$ or S, S' are each unions of additive cosets of $d\mathbb{Z}_q$ for $d \in \{2, 3, 4\}$ a divisor of q .*

If Σ is the event $\{A, B, C \subseteq E: |A| + |B| + |C| \in S \pmod{q}\}$, Σ' the same event with $S' \subseteq \mathbb{Z}_q \setminus S$ in place of S , and Δ the event that $A \triangle B, B \triangle C$ are both eulerian, then

$\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma' \mid \Delta)$ involves an evaluation of the Tutte polynomial at a single point only if $|S| = |S'|$ and $q \in \{2, 3, 4, 6\}$ or S and S' are each unions of additive cosets of $d\mathbb{Z}_q$ for $d \in \{2, 3, 4, 6\}$ a divisor of q .

Note that if S is a union of additive cosets of $d\mathbb{Z}_q$ then the event Σ is a congruence condition modulo d so these choices for S are herewith ignored.

Proof. Let $h = 1_S - 1_{S'}$. The only integers $q \geq 2$ for which $\phi(q) \leq 2$ are 2, 3, 4, 6. By Lemma 9 either we are in the case where \hat{h} is supported on an additive subgroup $d\mathbb{Z}_q$ or $\hat{h}(k) \neq 0$ for all units of \mathbb{Z}_q , of which there are $\phi(q)$. In the latter case only if $\phi(q) \leq 2$ is it the case that $\hat{f}(k)\hat{h}(k) = 0$ for $k \notin \{1, -1\}$. The former case by Lemma 9 reduces to the latter with q replaced by d . \square

The only choices for $q \geq 3$ and $S, S' \subseteq \mathbb{Z}_q$ are up to exceptions trivial by Corollary 10 given by the following theorem, whose proof is a simple matter of substituting in the expressions provided by Lemma 3 and Lemma 8.

Theorem 11. Let $q \in \{3, 4, 6\}$. Suppose $A, B, C \subseteq E$ are chosen uniformly at random and Δ is the event that $A \triangle B$ (and $B \triangle C$) is eulerian. Suppose further that $S, S' \subseteq \mathbb{Z}_q$ and $\text{supp}(\hat{1}_S - \hat{1}_{S'}) = \{1, -1\}$ (or possibly $\{1, -1, q/2\}$ for the third case of the following statement). If Σ is the event $|A| \pm |B| (+|C|) \in S \pmod{q}$, Σ' is the event $|A| \pm |B| (+|C|) \in S' \pmod{q}$ and $\text{Bias}(\Sigma) \neq \text{Bias}(\Sigma')$, then

$$\frac{\text{Bias}(\Sigma \mid \Delta) - \text{Bias}(\Sigma' \mid \Delta)}{\text{Bias}(\Sigma) - \text{Bias}(\Sigma')} = \begin{cases} 2^{r(G)} (1 + \cos \frac{2\pi}{q})^{-|E|} (\cos \frac{2\pi}{q})^{r(G)} (\cos \frac{2\pi}{q} - 1)^{n(G)} T(G; \frac{1}{\cos \frac{2\pi}{q}}, \frac{1 + \cos \frac{2\pi}{q}}{1 - \cos \frac{2\pi}{q}}), \\ 2^{r(G)} (1 + \cos \frac{2\pi}{q})^{-|E|} (\cos \frac{2\pi}{q} - 1)^{n(G)} T(G; \cos \frac{2\pi}{q}, \frac{\cos \frac{2\pi}{q} + 1}{\cos \frac{2\pi}{q} - 1}), \\ 2^{r(G)} (1 + \cos \frac{2\pi}{q})^{-|E|} (\cos \frac{2\pi}{q} - 1)^{n(G)} T(G; 2 \cos \frac{2\pi}{q} - 1, \frac{\cos \frac{2\pi}{q} + 1}{\cos \frac{2\pi}{q} - 1}), \end{cases}$$

according to

$$\Sigma = \begin{cases} \{A, B \subseteq E: |A| - |B| \in S \pmod{q}\}, \\ \{A, B \subseteq E: |A| + |B| \in S \pmod{q}\}, \\ \{A, B, C \subseteq E: |A| + |B| + |C| \in S \pmod{q}\}, \end{cases}$$

$$\Sigma' = \begin{cases} \{A, B \subseteq E: |A| - |B| \in S' \pmod{q}\}, \\ \{A, B \subseteq E: |A| + |B| \in S' \pmod{q}\}, \\ \{A, B, C \subseteq E: |A| + |B| + |C| \in S' \pmod{q}\}. \end{cases}$$

Taking q even, $|S| = q/2$ and $S' = \mathbb{Z}_q \setminus S$, Theorem 11 gives $\text{Bias}(\Sigma \mid \Delta)/\text{Bias}(\Sigma)$ as a Tutte polynomial evaluation.

So what choices of S and S' fulfil the conditions of Theorem 11? The answer is to be found in the theorems of Sections 2.3.1 to 2.3.4, which are immediate corollaries of Lemmas 3 and 8 and Theorem 11.

2.3.1. Evaluations for $q = 2$

Proposition 12. Suppose that $A, B \subseteq E$ are subgraphs of G chosen uniformly at random. Then the event that $|A| + |B|$ is even is correlated with the event Δ that $A \triangle B$ is eulerian as follows:

$$\text{Bias}(|A| + |B| \equiv 0 \pmod{2} \mid \Delta) = (-1)^{r(G)} T(G; -1, 0) = 2^{-k(G)} P(G; 2).$$

Proof. In Lemma 8 take $S = \{0\}$, $\Sigma = \{A, B \subseteq E: |A| + |B| \equiv 0 \pmod{2}\}$ and Δ the event that $A \triangle B$ is eulerian. Then, by the result of that lemma, $\text{Bias}(\Sigma \mid \Delta) = \hat{f}(0) + \hat{f}(1) - 1$, where $\hat{f}(0) = 1$ and $\hat{f}(1) = 2^{-n(G)} (-1)^{r(G)} 2^{r(G)} T(G; -1, 0) = 2^{-k(G)} P(G; 2)$. \square

Of course the correlation between parity and eulerian symmetric difference in Proposition 12 can be seen immediately by considering the identity

$$|A \triangle B| + 2|A \cap B| = |A| + |B|.$$

Eulerian subgraphs are all of even size if and only if G is bipartite (no odd cycles). Given the event Δ that $A \triangle B$ is eulerian the parity of $|A| + |B|$ must be even when G is bipartite. Otherwise, if G is not bipartite half the eulerian subgraphs are even, half odd, and so the parity of $|A| + |B|$ is equally likely to be even or odd given Δ .

For three subgraphs $A, B, C \subseteq E$ of G , if $A \triangle B, B \triangle C$ are eulerian then so is $C \triangle A$. From the identity

$$|A \triangle B| + |B \triangle C| + |C \triangle A| = 2(|A| + |B| + |C|) - 2(|A \cap B| + |B \cap C| + |C \cap A|),$$

it seems difficult to tell whether there might be any correlation between the event that $A \triangle B, B \triangle C$ are eulerian and some condition on $|A| + |B| + |C|$.

Theorem 13. Suppose $A, B, C \subseteq E$ are subgraphs of G chosen uniformly at random. Then the event that $|A| + |B| + |C|$ is even is uncorrelated with the event Δ that $A \triangle B, B \triangle C$ are eulerian, i.e.,

$$\text{Bias}(|A| + |B| + |C| \equiv 0 \pmod{2} \mid \Delta) = 0.$$

Proof. Take $\Sigma = \{A, B, C \subseteq E: |A| + |B| + |C| \equiv 0 \pmod{2}\}$ and Δ the event that $A \triangle B$ and $B \triangle C$ are both eulerian. By Lemma 8, $\text{Bias}(\Sigma \mid \Delta) = \hat{f}(0) + \hat{f}(1) - 1$, where $\hat{f}(0) = 1$ and $\hat{f}(1) = 0$. \square

However, we shall see that the residue of $|A| + |B| + |C|$ modulo 3, 4 and 6 does have a bearing on the event that $A \triangle B, B \triangle C$ are eulerian.

2.3.2. Evaluations for $q = 3$

The evaluations of the Tutte polynomial obtained for $q = 3$ are, unlike the cases $q = 2, 4$ and 6, at points without other more familiar combinatorial interpretations.

Theorem 14. Let $A, B \subseteq E$ be chosen uniformly at random and let Δ be the event that $A \triangle B$ is eulerian. Then

$$\text{Bias}(|A| \equiv |B| + 1 \pmod{3} \mid \Delta) = \text{Bias}(|A| \equiv |B| + 2 \pmod{3} \mid \Delta)$$

and

$$\frac{\text{Bias}(|A| \equiv |B| \pmod{3} \mid \Delta) - \text{Bias}(|A| \equiv |B| + 1 \pmod{3} \mid \Delta)}{\text{Bias}(|A| \equiv |B| \pmod{3}) - \text{Bias}(|A| \equiv |B| + 1 \pmod{3})}$$

$$= (-2)^{r(G)} 3^{n(G)} T\left(G; -2, \frac{1}{3}\right).$$

Proof. Lemma 8 with $S = \{1\}$ yields

$$\text{Bias}(|A| - |B| \equiv 1 \pmod{3} \mid \Delta) = \frac{2}{3} (1 + \hat{f}(1)e^{2\pi i/3} + \hat{f}(2)e^{4\pi i/3}) - 1,$$

where $f(\ell) = \mathbb{P}(|A| - |B| \equiv \ell \pmod{3} \mid \Delta)$. (We define a character χ on \mathbb{Z}_3 by setting $\chi(1) = e^{2\pi i/3}$, and the Fourier transform is defined by $\hat{f}(k) = f(0) + e^{4k\pi i/3} f(1) + e^{2k\pi i/3} f(2)$.)

Thus $\hat{1}_1 = 1_0 + e^{4\pi i/3} 1_1 + e^{2\pi i/3} 1_2$ and $\hat{1}_2 = \overline{\hat{1}_1}$.

With $S = \{2\}$ the same lemma yields

$$\text{Bias}(|A| - |B| \equiv 2 \pmod{3} \mid \Delta) = \frac{2}{3} (1 + \hat{f}(1)e^{4\pi i/3} + \hat{f}(2)e^{2\pi i/3}) - 1.$$

Since Lemma 8 also tells us that $\hat{f}(1) = \hat{f}(2)$, the first statement of the theorem is established.

By Theorem 11 with $S = \{0\}$, $S' = \{1\}$ (for which $\hat{1}_S - \hat{1}_{S'} = (1 - e^{4\pi i/3})1_1 + (1 - e^{2\pi i/3})1_2$), $\Sigma = \{A, B \subseteq E: |A| - |B| \equiv 0 \pmod{3}\}$ and $\Sigma' = \{A, B \subseteq E: |A| - |B| \equiv 1 \pmod{3}\}$,

$$\frac{\text{Bias}(|A| - |B| \equiv 0 \pmod{3} \mid \Delta) - \text{Bias}(|A| - |B| \equiv 1 \pmod{3} \mid \Delta)}{\text{Bias}(|A| - |B| \equiv 0 \pmod{3}) - \text{Bias}(|A| - |B| \equiv 1 \pmod{3})}$$

$$= 2^{r(G)} \left(\frac{1}{2}\right)^{-|E|} \left(-\frac{1}{2}\right)^{r(G)} \left(-\frac{3}{2}\right)^{n(G)} T\left(G; -2, \frac{1}{3}\right). \quad \square$$

Theorem 15. Let $A, B \subseteq E$ be chosen uniformly at random and let Δ be the event that $A \triangle B$ is eulerian. Then

$$\text{Bias}(|A| + |B| \equiv |E| + 1 \pmod{3} \mid \Delta) = \text{Bias}(|A| + |B| \equiv |E| + 2 \pmod{3} \mid \Delta)$$

and

$$\frac{\text{Bias}(|A| + |B| \equiv |E| \pmod{3} \mid \Delta) - \text{Bias}(|A| + |B| \equiv |E| + 1 \pmod{3} \mid \Delta)}{\text{Bias}(|A| + |B| \equiv |E| \pmod{3}) - \text{Bias}(|A| + |B| \equiv |E| + 1 \pmod{3})}$$

$$= 4^{r(G)} (-3)^{n(G)} T\left(G; -\frac{1}{2}, -\frac{1}{3}\right).$$

Proof. Lemma 8 with $S = \{|E| + 1\}$ yields

$$\text{Bias}(|A| + |B| \equiv |E| + 1 \pmod{3} \mid \Delta)$$

$$= \frac{2}{3} (1 + \hat{f}(1)e^{2\pi i(|E|+1)/3} + \hat{f}(2)e^{4\pi i(|E|+1)/3}) - 1,$$

where $f(\ell) = \mathbb{P}(|A| + |B| \equiv \ell \pmod{3} \mid \Delta)$. With $S = \{|E| + 2\}$ the same lemma yields

$$\text{Bias}(|A| + |B| \equiv |E| + 2 \pmod{3} \mid \Delta)$$

$$= \frac{2}{3} (1 + \hat{f}(1)e^{2\pi i(|E|+2)/3} + \hat{f}(2)e^{4\pi i(|E|+2)/3}) - 1.$$

Lemma 8 tells us that $\hat{f}(1) = e^{2\pi i|E|/3} \hat{f}(2)$, and the first statement of the theorem follows with both biases equal to $\hat{f}(1)(e^{2\pi i(|E|+1)/3} + e^{2\pi i(|E|+2)/3}) - \frac{1}{3}$.

The second statement of the theorem follows from Theorem 11 upon taking $S = \{|E|\}$, $S' = \{|E| + 1\}$, $\Sigma = \{A, B \subseteq E: |A| + |B| \equiv |E| \pmod{3}\}$ and $\Sigma' = \{A, B \subseteq E: |A| + |B| \equiv |E| + 1 \pmod{3}\}$. \square

Note that by Eq. (3) at the beginning of this section, if G is eulerian then the evaluations of the Tutte polynomial in Theorem 14 and Theorem 15 are equal. Indeed, $|E \setminus A| + |B| \equiv \pm 1 \pmod{3}$ if and only if $|A| - |B| \equiv |E| \mp 1 \pmod{3}$, and if G is eulerian then a subgraph A is eulerian if and only if its complement $E \setminus A$ is eulerian.

Theorem 16. *Let $A, B, C \subseteq E$ be chosen uniformly at random and let Δ be the event that $A \triangle B, B \triangle C$ are both eulerian. Then*

$$\text{Bias}(|A| + |B| + |C| \equiv 1 \pmod{3} \mid \Delta) = \text{Bias}(|A| + |B| + |C| \equiv 2 \pmod{3} \mid \Delta),$$

and

$$\begin{aligned} & \frac{\text{Bias}(|A| + |B| + |C| \equiv 0 \pmod{3} \mid \Delta) - \text{Bias}(|A| + |B| + |C| \equiv 1 \pmod{3} \mid \Delta)}{\text{Bias}(|A| + |B| + |C| \equiv 0 \pmod{3}) - \text{Bias}(|A| + |B| + |C| \equiv 1 \pmod{3})} \\ &= 4^{r(G)} (-3)^{n(G)} T\left(G; -2, -\frac{1}{3}\right). \end{aligned}$$

Proof. Lemma 8 with $S = \{1\}$ yields

$$\text{Bias}(|A| + |B| + |C| \equiv 1 \pmod{3} \mid \Delta) = \frac{2}{3} (\hat{f}(1)e^{2\pi i/3} + \hat{f}(2)e^{4\pi i/3}) - \frac{1}{3}$$

where $f(\ell) = \mathbb{P}(|A| + |B| + |C| \equiv \ell \pmod{3} \mid \Delta)$. Lemma 8 with $S = \{2\}$ yields

$$\text{Bias}(|A| + |B| + |C| \equiv 2 \pmod{3} \mid \Delta) = \frac{2}{3} (\hat{f}(1)e^{4\pi i/3} + \hat{f}(2)e^{2\pi i/3}) - \frac{1}{3}.$$

From Lemma 8 it is also found that $\hat{f}(1) = \hat{f}(2)$ and the first statement of the theorem follows.

Clearly then $\text{Bias}(|A| + |B| + |C| \equiv 0 \pmod{3}) \neq \text{Bias}(|A| + |B| + |C| \equiv 1 \pmod{3})$ and the second statement of the theorem results from Theorem 11 upon taking $S = \{0\}$, $S' = \{1\}$, $\Sigma = \{A, B, C \subseteq E: |A| + |B| + |C| \equiv 0 \pmod{3}\}$ and $\Sigma' = \{A, B, C \subseteq E: |A| + |B| + |C| \equiv 1 \pmod{3}\}$. \square

2.3.3. Evaluations for $q = 4$

Theorem 17. *Choosing $A, B \subseteq E$ uniformly at random, the event that $|A| - |B| \equiv 0$ or $1 \pmod{4}$ (i.e., $\lfloor \frac{|A| - |B|}{2} \rfloor$ is even) is correlated with the event Δ that $A \triangle B$ is eulerian:*

$$\frac{\text{Bias}(|A| - |B| \equiv 0, 1 \pmod{4} \mid \Delta)}{\text{Bias}(|A| - |B| \equiv 0, 1 \pmod{4})} = 2^{r(G)}.$$

Proof. In Lemma 3 take $S = \{0, 1\}$, for which $\hat{1}_S(k) = 1 + i^{-k}$, and calculate

$$\begin{aligned} & \text{Bias}(|A| - |B| \equiv 0, 1 \pmod{4}) \\ &= 2^{-1-|E|} [2^{|E|} \cdot 2 + 1 \cdot (1 + i) + 0 \cdot 0 + 1 \cdot (1 - i)] - 1 = 2^{-|E|}. \end{aligned}$$

That this is non-zero allows us to apply Theorem 11, in which we take $S = \{0, 1\}$, $S' = \{2, 3\} = \mathbb{Z}_4 \setminus S$, $\Sigma = \{A, B \subseteq E: |A| - |B| \equiv 0, 1 \pmod{4}\}$ and $\Sigma' = \{A, B \subseteq E:$

$|A| - |B| \equiv 2, 3 \pmod{4}$). This yields (recalling from the footnote on page 609 how to deal with division by zero in Tutte polynomial evaluations)

$$\frac{\text{Bias}(|A| - |B| \equiv 0, 1 \pmod{4} \mid \Delta) - \text{Bias}(|A| - |B| \equiv 2, 3 \pmod{4} \mid \Delta)}{\text{Bias}(|A| - |B| \equiv 0, 1 \pmod{4}) - \text{Bias}(|A| - |B| \equiv 2, 3 \pmod{4})} = 2^{r(G)}. \quad \square$$

Theorem 18. *Choosing $A, B \subseteq E$ uniformly at random, the event that $|A| + |B| \equiv 0$ or $1 \pmod{4}$ (i.e., $\lfloor \frac{|A|+|B|}{2} \rfloor$ is even) is correlated with the event Δ that $A \triangle B$ is eulerian in the following way:*

$$\frac{\text{Bias}(|A| + |B| \equiv 0, 1 \pmod{4} \mid \Delta)}{\text{Bias}(|A| + |B| \equiv 0, 1 \pmod{4})} = 2^{r(G)} F(G; 2).$$

Proof. In Lemma 3 take $S = \{0, 1\}$, for which $\widehat{1}_S(k) = 1 + i^{-k}$, and calculate

$$\begin{aligned} & \text{Bias}(|A| + |B| \equiv 0, 1 \pmod{4}) \\ &= 2^{-1-|E|} [2^{|E|} \cdot 2 + i^{-|E|} \cdot (1+i) + 0 \cdot 0 + i^{|E|} \cdot (1-i)] - 1 \\ &= (-1)^{\lfloor |E|/2 \rfloor} 2^{-|E|}. \end{aligned}$$

We can now apply Theorem 11, taking $S = \{0, 1\}$, $S' = \{2, 3\} = \mathbb{Z}_4 \setminus S$, $\Sigma = \{A, B \subseteq E: |A| + |B| \equiv 0, 1 \pmod{4}\}$ and $\Sigma' = \{A, B \subseteq E: |A| + |B| \equiv 2, 3 \pmod{4}\}$. The formula in Theorem 11 yields

$$\begin{aligned} & \frac{\text{Bias}(|A| + |B| \equiv 0, 1 \pmod{4} \mid \Delta) - \text{Bias}(|A| + |B| \equiv 2, 3 \pmod{4} \mid \Delta)}{\text{Bias}(|A| + |B| \equiv 0, 1 \pmod{4}) - \text{Bias}(|A| + |B| \equiv 2, 3 \pmod{4})} \\ &= 2^{r(G)} (-1)^{n(G)} T(G; 0, -1). \quad \square \end{aligned}$$

Theorem 18 says that if G is not eulerian then the event that $A \triangle B$ is eulerian removes from the parity of $\lfloor \frac{|A|+|B|}{2} \rfloor$ its original bias of $(-1)^{\lfloor |E|/2 \rfloor} 2^{-|E|}$ towards being even. Otherwise, when G is eulerian, this bias is accentuated by a factor of $2^{r(G)}$. Theorem 18 has a counterpart in Theorem 20 in Section 2.3.4 below.

Theorem 19. *Suppose $|E| \not\equiv 1 \pmod{4}$. Then, for $A, B, C \subseteq E$ chosen uniformly at random, the event that $|A| + |B| + |C| \equiv 0$ or $1 \pmod{4}$ (i.e., $\lfloor \frac{|A|+|B|+|C|}{2} \rfloor$ is even) is correlated with the event Δ that $A \triangle B$ and $B \triangle C$ are eulerian as follows:*

$$\frac{\text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4} \mid \Delta)}{\text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4})} = 2^{r(G)} (-1)^{n(G)} T(G; -1, -1).$$

If $|E| \equiv 1 \pmod{4}$ then

$$\text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4}) = 0 = \text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4} \mid \Delta).$$

Proof. In Lemma 3 take $q = 4$ and $S = \{0, 1\}$, for which $\widehat{1}_S(k) = 1 + i^{-k}$, and calculate

$$\begin{aligned} & \text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4}) \\ &= 2^{-1-\frac{3}{2}|E|} [2^{\frac{3}{2}|E|} \cdot 2 + i^{-\frac{3}{2}|E|} \cdot (1+i) + i^{\frac{3}{2}|E|} \cdot (1-i)] - 1 \\ &= 2^{-\frac{3}{2}|E|} \text{Re}[i^{\frac{3}{2}|E|} (1-i)] \end{aligned}$$

$$= \begin{cases} 2^{-3|E|/2} & |E| \equiv 0, 6 \pmod{8}, \\ 0 & |E| \equiv 1, 5 \pmod{8}, \\ -2^{-3|E|/2} & |E| \equiv 2, 4 \pmod{8}, \\ 2^{(1-3|E|)/2} & |E| \equiv 3 \pmod{8}, \\ -2^{(1-3|E|)/2} & |E| \equiv 7 \pmod{8}. \end{cases}$$

When $|E| \not\equiv 1 \pmod{4}$ we can apply Theorem 11 and the result follows by a straightforward calculation.

When $\text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4}) = 0$ we use Lemma 8 with $q = 4$, $S = \{0, 1\}$, to calculate that

$$\begin{aligned} & \text{Bias}(|A| + |B| + |C| \equiv 0, 1 \pmod{4} \mid \Delta) \\ &= 2 \operatorname{Re} \left[2^{-n(G) - \frac{1}{2}|E|} i^{\frac{3}{2}|E|} (-1)^{n(G)} T(G; -1, -1)(1 - i) \right] \\ &= 0 \quad \text{when } |E| \equiv 1 \pmod{4}. \quad \square \end{aligned}$$

From [14], $2^{r(G)}(-1)^{n(G)}T(G; -1, -1) = (-2)^{r(G) + \dim(\mathcal{C}_2 \cap \mathcal{C}_2^\perp)}$, where $\mathcal{C}_2 \cap \mathcal{C}_2^\perp$ is the bicycle space of G , comprising subgraphs which are both eulerian and bipartite.

2.3.4. Evaluations for $q = 6$

When $q = 6$ and $S = \{0, 1, 2\}$ the expression $\text{Bias}(\Sigma \mid \Delta) / \text{Bias}(\Sigma)$ is only equal to an evaluation of the Tutte polynomial at a single point when $\Sigma = \{A, B, C \subseteq E : |A| + |B| + |C| \equiv 0, 1, 2 \pmod{q}\}$. This is due to the formula for $\text{Bias}(\Sigma \mid \Delta)$ given by Lemma 8 and the fact that $\text{supp}(\widehat{1}_S) = \{0, \pm 1, 3\}$. For example, when Σ is the event $\{A, B \subseteq E : |A| + |B| \equiv 0, 1, 2 \pmod{6}\}$, evaluations of the Tutte polynomial at the two points $(\frac{1}{2}, -3)$ and $(0, -1)$ would be involved.

Theorem 20. Suppose that $A, B, C \subseteq E$ are chosen uniformly at random and Δ is the event that A, B, C have pairwise eulerian differences. Then the event that $|A| + |B| + |C| \equiv 0, 1, 2 \pmod{6}$ (i.e., $\lfloor \frac{|A|+|B|+|C|}{3} \rfloor$ is even) is correlated with Δ as follows:

$$\frac{\text{Bias}(|A| + |B| + |C| \equiv 0, 1, 2 \pmod{6} \mid \Delta)}{\text{Bias}(|A| + |B| + |C| \equiv 0, 1, 2 \pmod{6})} = 3^{-|E|} 4^{r(G)} F(G; 4).$$

Proof. In Lemma 3 take $q = 6$ and $S = \{0, 1, 2\}$, for which we calculate that $\widehat{1}_S = 3\mathbf{1}_0 + \mathbf{1}_3 - 2e^{2\pi i/3}\mathbf{1}_1 - 2e^{-2\pi i/3}\mathbf{1}_5$. By Lemma 3,

$$\text{Bias}(|A| + |B| + |C| \equiv 0, 1, 2 \pmod{6}) = \frac{2}{6} [\hat{g}(1) \overline{\widehat{1}_S(1)} + \hat{g}(5) \overline{\widehat{1}_S(5)}] = \frac{2}{3} \operatorname{Re} [\hat{g}(1) \overline{\widehat{1}_S(1)}]$$

where $\hat{g}(1) = 2^{-\frac{3}{2}|E|} e^{-2\pi i \frac{3}{2}|E|/6} (1 + \cos \frac{2\pi}{6})^{|E|} = 2^{-\frac{3}{2}|E|} e^{-|E|\pi i/2} (1 + \frac{1}{2})^{\frac{3}{2}|E|}$. Hence

$$\begin{aligned} \text{Bias}(|A| + |B| + |C| \equiv 0, 1, 2 \pmod{6}) &= \frac{2}{3} \operatorname{Re} \left[e^{-\pi i |E|/2} \left(\frac{3}{4} \right)^{\frac{3}{2}|E|} (2e^{-2\pi i/3}) \right] \\ &= \left(\frac{3}{4} \right)^{\frac{3}{2}|E|-1} \operatorname{Re} [i^{-|E|} e^{-2\pi i/3}] \neq 0 \end{aligned}$$

and we can apply Theorem 11 from which the result follows by routine calculation. \square

3. A parity criterion for proper vertex colourings

In the final three sections of this article we need some further identities from finite Fourier analysis for complete weight enumerators (which include Hamming weight enumerators as specialisations).

Let Q be a commutative ring with a generating character such as \mathbb{Z}_q or \mathbb{F}_q , and let f be a function $f: Q \rightarrow \mathbb{C}$.

The complete weight enumerator of a subset S of Q^E is defined by

$$\text{cwe}(S; f) = \sum_{x \in S} \prod_{e \in E} f(x_e).$$

When $f = t1_0 + 1_{Q \setminus 0}$ the complete weight enumerator is the Hamming weight enumerator $\text{hwe}(S; t)$. The MacWilliams duality theorem for complete weight enumerators is a consequence of the Poisson summation formula and states that when S is a Q -submodule of Q^E

$$\text{cwe}(S; f) = \frac{1}{|S^\perp|} \text{cwe}(S^\perp; \hat{f}). \quad (9)$$

The following generalises the first two identities of Lemma 1 and is proved for example in [7].

Lemma 21. Let Q^E be a commutative ring with a generating character. For Q -submodule S of Q^E and functions $f, g: Q \rightarrow \mathbb{C}$,

$$\sum_{S+z \in Q^E/S} \text{cwe}(S+z; f) \overline{\text{cwe}(S+z; g)} = \frac{1}{|S^\perp|} \text{cwe}(S^\perp; \hat{f} \cdot \bar{\hat{g}}).$$

Let \mathcal{C} be the set of Q -flows of G and its orthogonal \mathcal{C}^\perp the set of Q -tensions of G . A partial order \leq on Q^E is defined by $x \leq y$ if and only if $x_e \in \{0, y_e\}$ for all $e \in E$. (For $Q = \mathbb{F}_2$ the order \leq is set inclusion.) This makes the poset on Q^E the direct product of the poset P on Q defined by setting 0 below all the non-zero elements of Q and all pairs of non-zero elements incomparable. Thus the Möbius function of the poset $P^E = (Q^E, \leq)$ is defined by $\mu(x, y) = (-1)^{|y|-|x|}$. (See, for example, [16] for background on posets.) For a function $f: Q^E \rightarrow \mathbb{C}$, define $\mu f: Q^E \rightarrow \mathbb{C}$ by

$$\mu f(y) = \sum_{x \leq y} \mu(x, y) f(x) = \sum_{x \leq y} (-1)^{|y|-|x|} f(x).$$

Lemma 22. Let Q be a ring with a generating character χ . If \mathcal{C} is a Q -submodule of Q^E and \mathcal{C}^\perp its orthogonal space then

$$\mu 1_{\mathcal{C}}(y) = \frac{1}{|\mathcal{C}^\perp|} \sum_{x \in \mathcal{C}^\perp} \prod_{e \in E} (\chi(x_e y_e) - 1).$$

Proof.

$$\mu 1_{\mathcal{C}}(y) = \sum_{x \in \mathcal{C}} \prod_{e \in E} (1_{y_e} - 1_0)(x_e) = \frac{1}{|\mathcal{C}^\perp|} \sum_{x \in \mathcal{C}^\perp} (\bar{\chi}_{y_e} - 1)(x_e),$$

the latter equality by identity (9), and since the left-hand side is real $\bar{\chi}_{y_e}(x_e)$ can be replaced by its conjugate $\chi_{y_e}(x_e) = \chi(x_e y_e)$. \square

Lemma 23.

$$\sum_{y: \forall e \in E \ y_e \neq 0} \mu 1_{\mathcal{C}}(y) = (-1)^{|E|} |\mathcal{C}| \text{hwe}(\mathcal{C}^\perp; 0).$$

Proof.

$$\sum_{y: \forall e \in E \ y_e \neq 0} \mu 1_{\mathcal{C}}(y) = \sum_{y: \forall e \in E \ y_e \neq 0} \sum_{x \leq y, x \in \mathcal{C}} (-1)^{|y|-|x|} = \sum_{x \in \mathcal{C}} (q-1)^{|E|-|x|} (-1)^{|E|-|x|},$$

reversing the order of summation and using $\#\{y \in (Q \setminus 0)^E : x \leq y\} = (q-1)^{|E|-|x|}$, whence

$$\sum_{y: \forall e \in E \ y_e \neq 0} \mu 1_{\mathcal{C}}(y) = \sum_{x \in \mathcal{C}} (1-q)^{|E|-|x|} = \text{hwe}(\mathcal{C}; 1-q) = \frac{(-q)^{|E|}}{|\mathcal{C}^\perp|} \text{hwe}(\mathcal{C}^\perp; 0),$$

and finally $q^{|E|}/|\mathcal{C}^\perp| = |\mathcal{C}|$. \square

The following is a variation on, and mild generalisation of, Theorem 1.2 in [12].

Corollary 24. Suppose G is a graph and Q is a ring of order q with a generating character. Let \mathcal{C} be the set of Q -flows of G and \mathcal{C}^\perp the set of Q -tensions of G . Then $P(G; q) \neq 0$ if and only if there exists $y \in (Q \setminus 0)^E$ such that $\mu 1_{\mathcal{C}}(y) \neq 0$, i.e., such that

$$\sum_{x \leq y, x \in \mathcal{C}} (-1)^{|x|} \neq 0.$$

Proof. From Lemma 22, if $\chi(x_e y_e) \neq 1$ for all $e \in E$ then $x_e \neq 0$ for all $e \in E$. The converse follows from Lemma 23. \square

A dual to Corollary 24 giving a criterion for $F(G; q) \neq 0$ results by taking \mathcal{C} to be the set of Q -tensions. Corollary 24 was proved for $q = 3$ and generalised in a different direction by Alon and Tarsi [2] by considering \mathbb{Z}_q -flows taking values in $\{0, \pm 1\}$ only. The latter for q greater than the maximum degree of G are in bijective correspondence with partial eulerian orientations of G (and for $q = 3$ the same is true for 4-regular graphs). See also [17] and Section 5 below.

From Corollary 24 comes the familiar fact that $P(G; 2) \neq 0$ if and only if the number of eulerian subgraphs of G of even size differs from those with odd size. More interestingly, $P(G; 4) \neq 0$ if and only if there is $y \in \mathbb{F}_4^E$ such that $\mu 1_{\mathcal{C}}(y) \neq 0$, i.e., the difference between the number of \mathbb{F}_4 -flows $\leq y$ of even support size and those $\leq y$ of odd support size is non-zero, where in this case

$$\begin{aligned} \mu 1_{\mathcal{C}}(y) &= 4^{-r(G)} (-2)^{|E|} \#\{x \in \mathcal{C}^\perp : \forall e \in E \ x_e \notin \{0, y_e\}\} \\ &= (-2)^{|E|-2|V|} \#\{z \in \mathbb{F}_4^V : \forall uv \in E \ z_u + z_v \notin \{0, y_e\}\}. \end{aligned}$$

It may be that $\mu 1_{\mathcal{C}}(y) = 0$ for some $y \in (\mathbb{F}_4^\times)^E$ even though $P(G; 4) \neq 0$, since it may be impossible to avoid hitting the value y_e for some edge e in any nowhere-zero \mathbb{F}_4 -tension x of G . (For example, the triangle K_3 and $y_e = 1$ for each edge e .)

Similarly $P(G; 4) \neq 0$ if and only if for some $y \in (\mathbb{Z}_4 \setminus 0)^E$ there is a disparity between the number of \mathbb{Z}_4 -flows $\leq y$ of even support size and those \mathbb{Z}_4 -flows $\leq y$ of odd support size, and here

$$\mu 1_{\mathcal{C}}(y) = 4^{-r(G)} \sum_{x \in \mathcal{C}^\perp} \prod_{e \in E} (i^{x_e y_e} - 1).$$

If $y_e = 2$ then in order for $x \in \mathcal{C}^\perp$ to contribute a non-zero term it is necessary that $x_e \notin \{0, 2\}$. This too may not be possible for some y (consider K_3 again with $y_e = 2$ for each edge).

A translation of Corollary 24 for $Q = \mathbb{F}_4$ into the language of correlations between events involving parity and eulerian subgraphs runs as follows.

Theorem 25. *Suppose $X, Y, Z \subseteq E$ partition the edges of G into three sets (not all of which need be non-empty). Choosing $A \subseteq X$, $B \subseteq Y$ and $C \subseteq Z$ uniformly at random, let Σ be the event that $|A| + |B| + |C| \equiv 0 \pmod{2}$ and Γ the event that $A \cup C$ and $C \cup B$ are both eulerian.*

Then $\text{Bias}(\Sigma \mid \Gamma) \neq 0$ for some tripartition $\{X, Y, Z\}$ of E if and only if $P(G; 4) \neq 0$.

Note that in contrast to the event Δ of Section 2, it does not follow that if $A \cup C$, $C \cup B$ are eulerian then $A \cup B$ is eulerian. Also, note that $\text{Bias}(\Sigma) = 0$ for any choice of X, Y, Z .

Proof. We use Corollary 24 to show the auxiliary result that $P(G; 4) \neq 0$ if and only if there exist $X, Y \subseteq E$ with $X \cup Y = E$ and

$$\sum_{\substack{\text{eulerian } A \subseteq X, B \subseteq Y \\ A \Delta B \subseteq X \Delta Y}} (-1)^{|A \cup B|} \neq 0. \quad (10)$$

We then take $A \setminus B$, $B \setminus A$ in (10) for the A and B of the theorem, $X \setminus Y$, $Y \setminus X$ in (10) for the X and Y of the theorem, and finally set $C = A \cap B$ and $Z = X \cap Y$. This is enough to prove the theorem as stated, for (10), with $|(A \cup C) \cup (C \cup B)| = |A| + |B| + |C|$, is now the assertion that

$$\mathbb{P}(\Sigma \cap \Gamma) - \mathbb{P}(\bar{\Sigma} \cap \Gamma) = 2^{-3|E|} \sum_{\text{eulerian } A \subseteq X, B \subseteq Y, C \subseteq Z} (-1)^{|A| + |B| + |C|} \neq 0.$$

Let $x, y \in \mathbb{F}_2^E$ be the indicator vectors of X, Y and $z = (x, y) \in \mathbb{F}_2^E \times \mathbb{F}_2^E \cong \mathbb{F}_4^E$. Define a partial order \leq on \mathbb{F}_4^E by setting $d \leq z$ if and only if $d_e \in \{0, z_e\}$. Then $d = (a, b) \leq z = (x, y)$ if and only if $A \subseteq X$, $B \subseteq Y$ and $A \Delta B \subseteq X \Delta Y$. Note that $z_e \neq 0$ for all $e \in E$ if and only if $X \cup Y = E$. Denote by $|d|$ the Hamming weight of $d \in \mathbb{F}_4^E \cong \mathbb{F}_2^E \times \mathbb{F}_2^E$. If $d = (a, b)$ for $a, b \in \mathbb{F}_2^E$ the indicator vectors of $A, B \subseteq E$ then $|d| = |A \cup B|$.

Then an equivalent statement to (10) in terms of the space of \mathbb{F}_4 -flows $\mathcal{C}_4 \cong \mathcal{C}_2 \times \mathcal{C}_2$ is that $P(G; 4) \neq 0$ if and only if there exists $z \in (\mathbb{F}_4^\times)^E$ such that

$$\sum_{d \in \mathcal{C}_4, d \leq z} (-1)^{|d|} \neq 0.$$

This is the assertion of Corollary 24. \square

Theorem 25 is related to the criterion for $P(G; 4) \neq 0$ that G be covered by two bipartite subgraphs $X \cup Y, Y \cup Z$. Given the latter are bipartite, if $A \cup C \subseteq X \cup Z$ is eulerian and $C \cup B \subseteq Z \cup Y$ is eulerian then $|A \cup C|$ and $|C \cup B|$ are both even so that $|A| + |B|$ is also even. However, a bias in $|A| + |B| \pmod{2}$ does not imply a bias in $|A| + |B| + |C| \pmod{2}$.

4. Cubic graphs and triangulations

In this section we use MacWilliams duality theorem (9) for complete weight enumerators to derive a correlation criterion for the existence of a proper vertex 4-colouring of a triangulation.

Theorem 26. Let $\omega = e^{2\pi i/3}$ and let $\psi: \mathbb{F}_4 \rightarrow \{0, 1, \omega, \omega^2\}$ be a non-trivial Dirichlet character (multiplicative, and $\psi(0) = 0$). For a graph G with space of \mathbb{F}_4 -flows \mathcal{C}_4 and space of \mathbb{F}_4 -tensions \mathcal{C}_4^\perp ,

$$\sum_{z \in \mathcal{C}_4} \prod_{e \in E} \psi(z_e) = 2^{n(G)-r(G)} \sum_{z \in \mathcal{C}_4^\perp} \prod_{e \in E} \psi(z_e). \quad (11)$$

In other words

$$\sum_{\substack{\text{eulerian } A, B \subseteq E \\ A \cup B = E}} \omega^{|A| - |B|} = 2^{n(G)-r(G)} \sum_{\substack{\text{cutsets } A, B \subseteq E \\ A \cup B = E}} \omega^{|A| - |B|}. \quad (12)$$

In particular, if $G = (V, E)$ is a cubic graph then

$$2^{r(G)-n(G)} F(G; 4) = \# \left\{ z \in \mathcal{C}_4^\perp : \prod_{e \in E} z_e = 1 \right\} - \frac{1}{2} \# \left\{ z \in \mathcal{C}_4^\perp : \prod_{e \in E} z_e \in \{\omega, \omega^2\} \right\}. \quad (13)$$

Proof. We begin by noting that, since $z \in \mathcal{C}_4$ if and only if $\bar{z} \in \mathcal{C}_4$ and $\psi(\bar{z}_e) = \overline{\psi(z_e)}$, Eqs. (11) and (12) are between real numbers (in fact rational integers). Also, since $\prod_{e \in E} \psi(z_e) \neq 0$ if and only if z is nowhere-zero, i.e., $z_e \neq 0$ for all $e \in E$, the range of the summations in Eq. (11) is restricted to nowhere-zero \mathbb{F}_4 -flows on the left and nowhere-zero \mathbb{F}_4 -tensions on the right.

Identify \mathbb{F}_4 with its image $\{0, 1, \omega, \omega^2\}$ under $\psi: \mathbb{F}_4 \rightarrow \mathbb{C}$, i.e., ψ is defined by $\psi = 1_1 + \omega 1_\omega + \omega^2 1_{\omega^2}$. It is easily calculated that $\hat{\psi} = 2\overline{\psi}$. By the MacWilliams duality formula (9),

$$\sum_{z \in \mathcal{C}_4} \prod_{e \in E} \psi(z_e) = \text{cwe}(\mathcal{C}_4; \psi) = \frac{1}{|\mathcal{C}_4^\perp|} \text{cwe}(\mathcal{C}_4^\perp; \hat{\psi}) = 4^{-r(G)} 2^{|E|} \text{cwe}(\mathcal{C}_4^\perp, \overline{\psi}).$$

Since the sums in this equation are real, the function $\overline{\psi}$ can be replaced by its conjugate ψ , and this establishes Eq. (11) of the theorem.

The second statement (12) is a straight translation of (11) into different language. Using the isomorphism of additive groups $\mathbb{F}_4^E \cong \mathbb{F}_2^E \times \mathbb{F}_2^E$, an element $z \in \mathbb{F}_4^E$ may be written $z = (x, y)$ with $x, y \in \mathbb{F}_2^E$ indicator vectors for subsets $A, B \subseteq E$ respectively. The property that z is nowhere-zero translates to the property that $A \cup B = E$ and the condition $z \in \mathcal{C}_4$ translates to the condition that A and B are both eulerian. Similarly, the condition $z \in \mathcal{C}_4^\perp$ translates to the condition that A and B are cutsets.

For the final statement (13) we use the property that a cubic graph has a cutset double cover comprising the three-edge stars at each vertex. For vertex $v \in V$, the three edges $\{e, f, g\}$ incident with v form a star, and each edge occurs exactly twice amongst the $|V|$ stars, since each edge is adjacent to two distinct vertices. It follows that $\psi(z_e z_f z_g) \in \{0, 1\}$ for each star $\{e, f, g\}$ and $z \in \mathcal{C}_4$, due to the fact that if a sum of three non-zero elements of \mathbb{F}_4 is equal to 0 then their product is 1. Thus we see that

$$\sum_{z \in \mathcal{C}_4} \prod_{\text{stars } \{e, f, g\}} \psi(z_e z_f z_g) = F(G; 4).$$

On the other hand, by the double cover property of the collection of stars,

$$\sum_{z \in \mathcal{C}_4} \prod_{\text{stars } \{e, f, g\}} \psi(z_e z_f z_g) = \sum_{z \in \mathcal{C}_4} \prod_{e \in E} \psi(z_e)^2,$$

and $\psi(z_e)^2 = \bar{\psi}(z_e)$. Using Eq. (11), in which ψ is interchangeable with its conjugate $\bar{\psi}$, this establishes that

$$F(G; 4) = 2^{n(G)-r(G)} \sum_{z \in \mathcal{C}_4^\perp} \prod_{e \in E} \psi(z_e),$$

and Eq. (13) is just another way of writing this. \square

We finish this section by interpreting identity (13) in its dual form in terms of the bias of events in a uniform probability space. The dual notion of a cutset double cover is a cycle double cover. If G is a plane cubic graph then its planar dual G^* is a plane triangulation and just as G has a cutset double cover by three-edge stars (at vertices) so G^* has a cycle double cover by triangles (faces).

Theorem 27. *Suppose G is a graph that has a cycle double cover by triangles and suppose that Γ is the event that $A, B \subseteq E$ are eulerian and $A \cup B = E$. Then, choosing $A, B \subseteq E$ uniformly at random,*

$$\text{Bias}(|A| \equiv |B| + 1 \pmod{3} \mid \Gamma) = \text{Bias}(|A| \equiv |B| + 2 \pmod{3} \mid \Gamma)$$

and

$$\frac{\text{Bias}(|A| \equiv |B| \pmod{3} \mid \Gamma) - \text{Bias}(|A| \equiv |B| + 1 \pmod{3} \mid \Gamma)}{\text{Bias}(|A| \equiv |B| \pmod{3}) - \text{Bias}(|A| \equiv |B| + 1 \pmod{3})} = \frac{2^{3|E|-2|V|} P(G; 4)}{F(G; 4)}.$$

Proof. The left-hand sum in Eq. (12) of Theorem 26 has the following interpretation:

$$\begin{aligned} & 2^{-2|E|} \sum_{\substack{\text{eulerian } A, B \subseteq E \\ A \cup B = E}} \omega^{|A|-|B|} \\ &= \mathbb{P}(|A| \equiv |B| \pmod{3} \cap \Gamma) \\ & \quad + \omega \mathbb{P}(|A| \equiv |B| + 1 \pmod{3} \cap \Gamma) + \omega^2 \mathbb{P}(|A| \equiv |B| + 2 \pmod{3} \cap \Gamma) \\ &= \mathbb{P}(|A| \equiv |B| \pmod{3} \cap \Gamma) \\ & \quad + \omega^2 \mathbb{P}(|A| \equiv |B| + 1 \pmod{3} \cap \Gamma) + \omega \mathbb{P}(|A| \equiv |B| + 2 \pmod{3} \cap \Gamma), \end{aligned} \quad (14)$$

the latter equality since, as remarked in the proof of Theorem 26, the sum we started with is real. Hence

$$\mathbb{P}(|A| \equiv |B| + 1 \pmod{3} \cap \Gamma) = \mathbb{P}(|A| \equiv |B| + 2 \pmod{3} \cap \Gamma). \quad (15)$$

By definition of Γ and since G has a cycle double cover by triangles, $\mathbb{P}(\Gamma) = 2^{-2|E|} F(G; 4) \neq 0$. Dividing Eq. (15) by $\mathbb{P}(\Gamma)$ yields the first statement of the theorem.

Equation (15) together with the identity developed in (14) has the consequence that, in the notation of Theorem 26,

$$2^{-2|E|} \sum_{z \in \mathcal{C}_4^\perp} \prod_{e \in E} \psi(z_e) = \mathbb{P}(|A| \equiv |B| \pmod{3} \cap \Gamma) - \mathbb{P}(|A| \equiv |B| + 1 \pmod{3} \cap \Gamma). \quad (16)$$

By Eq. (11) of Theorem 26, in which $\prod_{e \in E} \psi(z_e) \in \{0, 1\}$ for $z \in \mathcal{C}_4^\perp$ since G has a cycle double cover by triangles, and Eq. (16),

$$\begin{aligned} & \mathbb{P}(|A| \equiv |B| \pmod{3} \cap \Gamma) - \mathbb{P}(|A| \equiv |B| + 1 \pmod{3} \cap \Gamma) \\ &= 2^{-2|E|} 2^{n(G)-r(G)} \sum_{z \in \mathcal{C}_4^\perp} \prod_{e \in E} \psi(z_e) = 2^{-2|E|} 2^{n(G)-r(G)} 4^{-k(G)} P(G; 4). \end{aligned}$$

Dividing this last equation through by $\mathbb{P}(\Gamma) = 2^{-2|E|} F(G; 4)$ gives

$$\mathbb{P}(|A| \equiv |B| \pmod{3} \mid \Gamma) - \mathbb{P}(|A| \equiv |B| + 1 \pmod{3} \mid \Gamma) = 2^{|E|-2|V|} P(G; 4)/F(G; 4),$$

i.e.,

$$\begin{aligned} & \text{Bias}(|A| \equiv |B| \pmod{3} \mid \Gamma) - \text{Bias}(|A| \equiv |B| + 1 \pmod{3} \mid \Gamma) \\ &= 2^{|E|-2|V|+1} P(G; 4)/F(G; 4). \end{aligned}$$

By Lemma 3 with $q = 3$ and $\Sigma = \{A, B \subseteq E : |A| - |B| \in S\}$, taking the difference between the cases $S = \{0\}$ and $S = \{1\}$ we obtain

$$\begin{aligned} & \text{Bias}(|A| - |B| \equiv 0 \pmod{3}) - \text{Bias}(|A| - |B| \equiv 1 \pmod{3}) \\ &= 3^{-1} 2^{1-|E|} \left[\left(1 - \frac{1}{2}\right)^{|E|} (1 - e^{2\pi i/3}) + \left(1 - \frac{1}{2}\right)^{|E|} (1 - e^{4\pi i/3}) \right] \\ &= 2^{1-2|E|}. \end{aligned}$$

The second statement of the theorem now results. \square

In particular, a graph G with a cycle double cover by triangles has $P(G; 4) \neq 0$ if and only if $\mathbb{P}(|A| \equiv |B| \pmod{3} \mid \Gamma) > \frac{1}{3}$, i.e., the event that A, B form an eulerian cover of G is positively correlated with $|A| \equiv |B| \pmod{3}$.

5. Eulerian subdigraphs of a 4-regular graph

In this final section we use MacWilliams duality (9) for complete weight enumerators and Lemma 21 to derive some further evaluations of the Tutte polynomial on H_3 similar in form to Theorem 27.

Take $q = 3$ and G a 4-regular graph (such as the line graph of a plane cubic graph), for which the space of \mathbb{F}_3 -flows has a natural identification with the set of eulerian partial orientations of G . In a partial orientation of a graph some edges may not be directed; in an *eulerian* partial orientation each vertex has the same number of incoming and outgoing directed edges.

A reference orientation γ of G is fixed. A partial orientation α is defined corresponding to a vector $x \in \mathbb{F}_3^E$ by making α direct an edge e the same way as γ if $x_e = +1$, making α reverse the direction of γ if $x_e = -1$, and leaving e undirected if $x_e = 0$. Given that G is 4-regular, if x is a \mathbb{F}_3 -flow then it defines an eulerian partial orientation α . If further x is nowhere-zero then α is an eulerian orientation of G . For orientations α, β , define $\alpha + \beta$ to be the partial orientation whose directed edges are those sharing the same direction in α and β . If $x, y \in \mathbb{F}_3^E$ define α, β relative to the base orientation γ of G then $\alpha + \beta$ is the partial orientation defined by $-(x + y)$.

Suppose an orientation α is chosen uniformly at random. Let Σ be the event that $|\alpha + \gamma| \equiv 0 \pmod{2}$. Clearly $\text{Bias}(\Sigma) = 0$. Let Γ be the event that α is an eulerian orientation. Then $\mathbb{P}(\Gamma) = 2^{-|E|} F(G; 3)$, since for the 4-regular graph G the number of eulerian orientations is the number of nowhere-zero 3-flows.

Finding $\text{Bias}(\Sigma \mid \Gamma)$ is a bit more difficult and in order to state a partial result on this we need some definitions. The *line graph* $L(H)$ of a graph H has vertices the edges of H and adjacent vertices e, f when e and f are incident in H . If H is embedded in an orientable surface, the *medial graph* $M(H)$ of H is the graph obtained by placing vertices at the edges of H and joining vertices e, f of $M(H)$ by an edge if they lie on incident edges e, f of H and it is possible to draw a line joining e and f without crossing any edges of H . (If edges e, f are incident with a vertex of degree 2 then they are joined by two edges, neither of which can be continuously transformed to the other without crossing an edge of H .) The medial graph $M(H)$ is 4-regular.

Suppose now that H is an orientably embedded cubic graph. Then $M(H)$ is an embedding of $L(H)$ in the same orientable surface as H . A vertex of H lies in the interior of a triangle of edges in $M(H)$, which we shall call a *black triangle* of $M(H)$ (on account of the standard white-black face colouring of the medial graph). When the edges of the black triangles of $M(H)$ are directed in a clockwise sense on the surface on which $M(H)$ is embedded, the resulting orientation of $M(H)$ is eulerian. (The clockwise direction traced by the edges of a black triangle of $M(H)$ corresponds to a clockwise orientation of the three edges at a vertex of H , called a vertex rotation in the embedding of H .)

Theorem 28. *Let G be the medial graph of a plane cubic graph and let γ be the orientation directing edges of G clockwise around black triangles. Then, choosing an orientation α of G uniformly at random, the event Σ that α agrees with γ on an even number of edges and the event Γ that α is eulerian have correlation given by*

$$\text{Bias}(\Sigma \mid \Gamma) = \frac{P(G; 3)}{F(G; 3)}.$$

Proof. Let \mathcal{C}_3 be the space of \mathbb{F}_3 -flows of G . Then

$$\begin{aligned} \mathbb{P}(\Gamma) \text{Bias}(\Sigma \mid \Gamma) &= \mathbb{P}(\Sigma \cap \Gamma) - \mathbb{P}(\bar{\Sigma} \cap \Gamma) \\ &= 2^{-|E|} \text{cwe}(\mathcal{C}_3; 1_1 - 1_{-1}) \\ &= 2^{-|E|} 3^{-r(G)} \text{cwe}(\mathcal{C}_3^\perp; (-3)^{\frac{1}{2}}(1_{-1} - 1_1)) \\ &= (-1)^{|V|} 2^{-|E|} 3^{k(G)} \text{cwe}(\mathcal{C}_3^\perp; 1_1 - 1_{-1}), \end{aligned}$$

using $|E| = 2|V|$ for 4-regular graph G . A result of Penrose⁴ says that for any nowhere-zero \mathbb{F}_3 -tension x of G (corresponding to an edge 3-colouring of the plane cubic graph H) we have

$$\prod_{e \in E} (1_1 - 1_{-1})(x_e) = (-1)^{\#\{e \in E: x_e = -1\}} = (-1)^{|V|}$$

when G has its fixed orientation γ clockwise around black triangles (or any other orientation β with $|\beta + \gamma|$ even). With $\mathbb{P}(\Gamma) = 2^{-|E|} F(G; 3) \neq 0$ the theorem is proved. \square

⁴ Penrose quoted this theorem (in a different formulation) in [13], in which he mentioned that his proof was too lengthy for inclusion. An elegant short proof has been given by Kaufmann [8]. See also [5, Theorem 3.1] for a generalisation and for further citations—for example Scheim [15] found the result independently and was the first to publish a proof.

It remains an open problem [5] to characterise those edge 3-colourable cubic graphs H for which the line graph $L(H)$ with a fixed orientation of its edges has the property that the number of nowhere-zero \mathbb{F}_3 -tensions of $L(H)$ with an even number of edges with value -1 differs from those with an odd number of edges with value -1 . The line graph $L(K_{3,3})$ does not have this property. The theorem of Penrose and Scheim is that when H is a planar cubic graph nowhere-zero \mathbb{F}_3 -tensions of $L(H)$ either all have an even number of edges with value -1 or all an odd number of such edges.

We move on now to choosing pairs of orientations of a 4-regular graph and formulate an analogue of Proposition 12 for eulerian orientations rather than the eulerian subgraphs of that proposition. Suppose α, β are orientations of G chosen uniformly at random. Let Σ be the event that $|\alpha + \beta|$ is even. Note that if β is another orientation then $|\alpha + \gamma|$ and $|\beta + \gamma|$ have the same parity if and only if $|\alpha + \beta|$ is even. We have $\text{Bias}(\Sigma) = 0$ as before. Let Γ be the event that $\alpha + \beta$ is an eulerian partial orientation of G , i.e., the consistently directed edges of α and β form an eulerian partial orientation of G .

Lemma 29. (Cf. [6, Corollary 5].) *The event Γ that orientations α, β of a 4-regular graph agree on an eulerian partial orientation has probability*

$$\mathbb{P}(\Gamma) = 4^{-|E|} T(G; 2, 4).$$

Proof. Let \mathcal{C}_3 be the space of \mathbb{F}_3 -flows of G . If orientations α, β are defined relative to the base orientation γ of G by the nowhere-zero vectors $x, -y \in \mathbb{F}_3^E$, then the event that $\alpha + \beta$ is eulerian coincides with the event that $x - y \in \mathcal{C}_3$, i.e., x, y belong to the same coset of \mathcal{C}_3 . Using Lemma 21,

$$\begin{aligned} \mathbb{P}(\Gamma) &= 4^{-|E|} \sum_{\mathcal{C}_3 + z \in \mathbb{F}_3^E / \mathcal{C}_3} \text{cwe}(\mathcal{C}_3 + z; 1_1, -1)^2 \\ &= 4^{-|E|} 3^{-r(G)} \text{cwe}(\mathcal{C}_3^\perp; (21_0 - 1_1, -1)^2) \\ &= 4^{-|E|} 3^{-r(G)} \text{hwe}(\mathcal{C}_3^\perp; 4) \\ &= 4^{-|E|} T(G; 2, 4). \quad \square \end{aligned}$$

This lemma leads us to our promised theorem.

Theorem 30. *Let G be a 4-regular graph and α, β two orientations of G chosen uniformly at random. Then the event Σ that α agrees with β on an even number of edges and the event Γ that α agrees with β in an eulerian partial orientation of G have correlation given by*

$$\text{Bias}(\Sigma \mid \Gamma) = \frac{3^{|E|-|V|} P(G; 3)}{T(G; 2, 4)}.$$

Proof. Using Lemma 21,

$$\begin{aligned} \mathbb{P}(\Gamma) \text{Bias}(\Sigma \mid \Gamma) &= \mathbb{P}(\Sigma \cap \Gamma) - \mathbb{P}(\bar{\Sigma} \cap \Gamma) \\ &= 4^{-|E|} \sum_{\mathcal{C}_3 + z \in \mathbb{F}_3^E / \mathcal{C}_3} |\text{cwe}(\mathcal{C}_3 + z; 1_1 - 1_{-1})|^2 \\ &= 4^{-|E|} 3^{-r(G)} \text{cwe}(\mathcal{C}_3^\perp; |(-3)^{\frac{1}{2}}(1_{-1} - 1_1)|^2) \\ &= 4^{-|E|} 3^{-r(G)} \text{cwe}(\mathcal{C}_3^\perp; 31_1, -1) = 4^{-|E|} 3^{n(G)} (-1)^{r(G)} T(G; -2, 0) \\ &= 4^{-|E|} 3^{|E|-|V|} P(G; 3). \end{aligned}$$

Lemma 29 now gives the result. \square

Acknowledgments

I am grateful to the referees for their patient reading of the first version of this article, for their helpful comments which led to many improvements, and for alerting me to the reference [9].

References

- [1] N. Alon, Combinatorial Nullstellensatz, *Combin. Probab. Comput.* 8 (1 & 2) (1999) 7–29.
- [2] N. Alon, M. Tarsi, Colorings and orientations of graphs, *Combinatorica* 12 (1992) 125–134.
- [3] N. Alon, M. Tarsi, A note on graph colorings and graph polynomials, *J. Combin. Theory Ser. B* 70 (1997) 197–201.
- [4] B. Bollobás, *Modern Graph Theory*, Springer, New York, 1998.
- [5] M.N. Ellingham, L. Goddyn, List edge colourings of some 1-factorable multigraphs, *Combinatorica* 16 (1996) 343–352.
- [6] A. Goodall, Some new evaluations of the Tutte polynomial, *J. Combin. Theory Ser. B* 96 (2006) 207–224.
- [7] A. Goodall, Fourier analysis on finite abelian groups: Some graphical applications, Chapter 7 in: G. Grimmett, C. McDiarmid (Eds.), *Combinatorics, Complexity, Chance: A Tribute to Dominic Welsh*, Oxford Lecture Ser. Math. Appl., vol. 34, Oxford Univ. Press, Oxford, 2007, pp. 103–129.
- [8] L. Kaufmann, Map coloring and the vector cross product, *J. Combin. Theory Ser. B* 48 (1990) 145–154.
- [9] J. Kung, Coboundary, flows and Tutte polynomials of matrices, *Ann. Combin.*, in press.
- [10] Y. Matiyasevich, Some probabilistic restatements of the four color conjecture, *J. Graph Theory* 46 (2004) 167–179.
- [11] O. Milenkovic, Support weight enumerators and coset weight distributions of isodual codes, *Des. Codes Cryptogr.* 35 (1) (2005) 81–109.
- [12] S. Onn, Nowhere-zero flow polynomials, *J. Combin. Theory Ser. A* 108 (2004) 205–215.
- [13] R. Penrose, Applications of negative dimensional tensors, in: D. Welsh (Ed.), *Combinatorial Mathematics and its Applications*, Academic Press, New York, London, 1971, pp. 221–244.
- [14] R.C. Read, P. Rosenstiehl, On the principal edge tripartition of a graph, *Ann. Discrete Math.* 3 (1978) 195–226.
- [15] D.E. Scheim, The number of edge 3-colorings of a planar cubic graph as a permanent, *Discrete Math.* 8 (1974) 377–382.
- [16] R. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge Univ. Press, 1997.
- [17] M. Tarsi, The graph polynomial and the number of proper vertex colorings, *Ann. Inst. Fourier* 49 (3) (1999) 1089–1093.
- [18] A. Terras, *Fourier Analysis on Finite Groups and Applications*, Cambridge Univ. Press, 1999.
- [19] B. van der Waerden, Die lange Reichweite der regelmässigen Atomanordnung in Mischkristallen, *Z. Physik* 118 (1941) 473.
- [20] D. Welsh, *Complexity: Knots, Colourings and Counting*, Cambridge Univ. Press, 1993.